PHD

Absolute stability results for infinite-dimensional discrete-time systems with applications to sampled-data integral control

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Award date:
2007

Awarding institution:
University of Bath

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for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

June 2007

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Summary

We derive, in an input-output setting, absolute stability results of Popov and circle-criterion-type for discrete-time non-linear feedback systems, where the linear part is the series interconnection of an l^2-stable shift-invariant linear system pre-compensated by an integrator. We apply this discrete-time stability theory to obtain input-output results on low-gain integral control of discrete-time systems in the presence of actuator and sensor non-linearities. This discrete-time theory is in turn applied to develop a low-gain sampled-data integral control strategy for tracking of constant reference signals in the context of L^2-stable shift-invariant linear systems subject to non-decreasing globally Lipschitz actuator non-linearities. It is shown that applying error feedback using a sampled-data integral controller ensures that the tracking error is asymptotically small in a certain sense, provided that (a) the transfer function of the linear system is holomorphic in a neighbourhood of 0, (b) the steady-state gain is positive, (c) the reference value is feasible in an entirely natural sense, and (d) the positive-valued (possibly time-varying) integrator gain is ultimately sufficiently small and not summable. Generalised as well as ideal samplers are considered together with zero-order hold. In the case of ideal sampling sensor non-linearities can also be included. The discrete-time and sampled-data input-output results are applied to infinite-dimensional state-space settings. Applications of the discrete-time stability theory to numerical linear multistep methods are also discussed.

Acknowledgements

I would like to thank the University of Bath Mathematics department for funding my PhD for the past three years. Many thanks go to Professor Hartmut Logemann for his tireless support and enthusiasm throughout the project. Thanks also to Dr Adrian Hill for his insight and helping to shape the project in a new direction.

I would like to thank my parents John and Pauline Coughlan for all their support, financial and otherwise, throughout the years. Also, thanks to the rest of my family, Joseph, Stephen and Kathleen, Maryanne, Patrick and Teresa. I would like to thank my Nan for all the Sundays we spent together and meals she cooked for me.

I was lucky to be surrounded by some wonderful office mates and friends during my time at Bath, many thanks go to Damo, Matt, Andy, Lorina, Doku and Jay. Outside of the office I would like to thank James and Jo, Barrie, Rachel and baby Kirin, Eugen, Ant, Phil and all the other Postgrads.

Thanks to Duncan for being such a wonderful friend and number one housemate and thanks for all the cake Clare!

The Maths and Computer Science Postgrad football team kept me sane most weeks thanks to all the guys who played in this team (league champions finally!!!!). I was also lucky enough to be a part of the Bath University Venturers Cricket Club thanks for all the wonderful weekends out in the sun (and rain!) Thanks to Liverpool Football Club for the heroics of Istanbul! YNWA!

There are many people in Coventry I would like to thank but the list would be too long, but thanks to Ben for his enthusiasm for the subject, Sam, John, Karl, Dan, Little Steve, Little Ben, Wiggy, Mike, Jim, Bomber, for being great friends and drinking partners! Most people are inspired by great school teachers, to this end I would like to thank Mr Colin Little.

Most of all, a very special thank you to Helen for her love and support.

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Chapter 1

Introduction

The term absolute stability refers to the study of the stability of an entire family of systems. Consider the feedback system shown in Figure 1.1, where L is a linear shift-invariant system and N is a (possibly time-varying) static non-linearity. For simplicity, we assume that L and N are 'scalar' systems, that is, L and N have only one input and one output channel. A sector condition for N is a condition of the form

$$a_1 v^2 \leq N(t,v)v \leq a_2 v^2, \quad \forall\ (t,v) \in \mathbb{R}_+ \times \mathbb{R}, \tag{1.1}$$

where $-\infty \leq a_1 \leq a_2 \leq \infty$ and at least one of the sector bounds a_1 and a_2 is finite. Standard examples of sector-bounded non-linearities are given by deadzone and saturation (see, for example, Figure 1.2), both of which arise naturally in control engineering. An absolute stability result for the feedback system shown in Figure 1.1 is a stability criterion in terms of the transfer function or the frequency response of the linear system L and the sector bounds a_1 and a_2 of the non-linearity N. Given a linear system L and sector data a_1 and a_2, an absolute stability criterion guarantees closed-loop stability for all non-linearities N satisfying the sector condition (1.1).

Absolute stability problems and their relations to positive-real conditions have played a prominent role in systems and control theory and permeate much of the classical and modern control literature, see Fliegner *et al.* [19], Haddad and Kapila [22], Halanay and Răsvan [23], Hu and Lin [28], Khalil [31], Szegö and Pearson [58], Vidyasagar [62] for the finite-dimensional case, and Corduneanu [3], Curtain *et al.* [9, 10], Curtain and Oostveen [11], Desoer and Vidyasagar [15], Logemann and Curtain [33], Logemann and Ryan [38], Vidyasagar [62] and Wexler [66, 67] for the infinite-dimensional case, to mention just a few references. In the finite-dimensional case there are a large number of results available in the literature, many of which have been obtained by Lyapunov techniques applied to state-space models with the so-called Kalman-Yakubovich-Popov (or positive-real) lemma playing a crucial role. In the infinite-dimensional case the literature

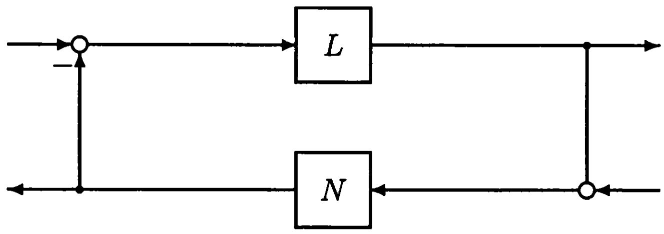

Figure 1.1: Feedback system with non-linearity

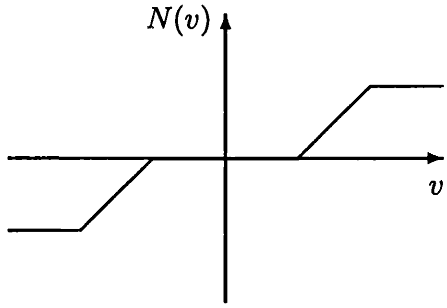

Figure 1.2: Non-linearity with saturation and deadzone

on absolute stability problems is dominated by input-output approaches.

Of particular importance in the context of static non-linearities, are absolute stability results of circle-criterion type and those of Popov-type, each applicable to the feedback system shown in Figure 1.1. In recent papers, Curtain, Logemann and Staffans [9, 10] presented continuous-time absolute stability results of Popov and circle-criterion type in infinite-dimensions. In [9] these results are applied in the context of integral control in the presence of input/output non-linearities. The conjunction of the series interconnection of a stable system pre-compensated by an integrator and non-linearities with possibly zero lower gain that is, $a_1 = 0$ in (1.1) (so-called critical cases of circle and Popov criteria) is a distinguishing feature of [9] and [10]. The approach taken in [9] and [10] is the input-output approach as opposed to the state-space approach.

While most of the available absolute-stability literature is devoted to continuous-time systems, there are still a considerable number of references which treat discrete-time systems; see, for example, [15], [23], [28], [58] and [62]. In this thesis we consider a discrete-time absolute stability problem for the feedback system shown in Figure 1.3. A unique feature of discrete-time integral control is the different choices of integrator J and J_0, where J is a strictly causal integrator, whereas integrator J_0 has direct feedthrough. The input-output operator G is assumed to be l^2-stable, linear, and shift-invariant. Consequently, G has a transfer function $\mathbf{G}$ which is analytic and bounded on the exterior of the closed unit disc in the complex plane. The (possibly time-varying) non-linearity φ is sector bounded with potentially zero lower gain. Corresponding to the two integrators J and J_0 we consider slightly different positive real conditions. Let $\varepsilon \geq 0$ and $q \geq 0$. For integrator J:

$$P + \frac{1}{2}\left[q\mathbf{G}(e^{i\theta}) + \frac{1}{e^{i\theta}-1}Q\mathbf{G}(e^{i\theta}) + q\mathbf{G}^*(e^{i\theta}) + \frac{1}{e^{-i\theta}-1}\mathbf{G}^*(e^{i\theta})Q^*\right] \geq \varepsilon I, \quad (1.2)$$

and for integrator J_0:

$$P + \frac{1}{2}\left[q\mathbf{G}(e^{i\theta}) + \frac{e^{i\theta}}{e^{i\theta}-1}Q\mathbf{G}(e^{i\theta}) + q\mathbf{G}^*(e^{i\theta}) + \frac{e^{-i\theta}}{e^{-i\theta}-1}\mathbf{G}^*(e^{i\theta})Q^*\right] \geq \varepsilon I, \quad (1.3)$$

where (1.2) and (1.3) hold for a.a $\theta \in (0, 2\pi)$, $P : U \to U$ denotes a linear, bounded, self-adjoint operator, $Q : U \to U$ denotes a linear, bounded, invertible operator and U denotes a Hilbert space.

In this thesis, the main motivation for studying discrete-time absolute stability problems is to develop an input-output theory of low-gain integral control of discrete-time systems and low-gain sampled-data integral control, in the presence of static input and/or output non-linearities.

The low-gain integral control problem has its roots in control engineering, where it is often required that the output y of a system tracks a constant reference signal r, that is, the error $e(t) := y(t) - r$ should be small in some sense for large t. It is well-known (see, for example, Davison [14], Lunze [44] and Morari [46]) that, for exponentially stable continuous-time shift-invariant linear finite-dimensional systems with positive steady-state gain (that is, $\mathbf{G}(0) > 0$), this can be achieved by feeding the error into an integrator with sufficiently small positive

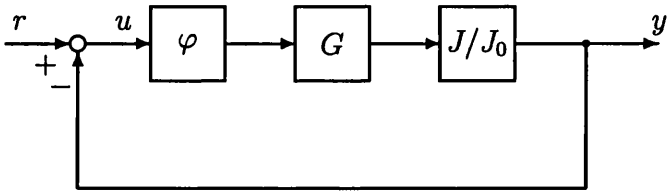

Figure 1.3: Discrete-time feedback system with choice of integrator

gain parameter and then closing the feedback loop.

There is a wealth of recent research on low-gain integral control, see, for example, Fliegner *et al.* [17, 18, 19], for the non-linear finite-dimensional case, Curtain *et al.* [9], Fliegner *et al.* [16], Logemann and Curtain [33], Logemann and Mawby [34], Logemann and Ryan [36, 37, 39], Logemann *et al.* [40], for the non-linear infinite-dimensional case and Logemann and Townley [42, 43] for the linear infinite-dimensional case. With the exception of [9] and [39] the above references adopt a state-space approach. Whilst most of the above references are concerned with continuous-time systems, there are a number of references which consider discrete-time and sampled-data systems, see for example, [17, 34, 37, 42].

We now give some additional details on the problems considered. Chapter 2 consists of some preliminaries required for the rest of the thesis. In Chapter 3, we discuss three key notions; the convolution of two sequences, the $\mathscr{Z}$-transform of a sequence, and transfer functions of bounded, linear, shift-invariant operators on l^2. We introduce the concepts of asymptotic steady-state gain, l^2-steady-state gain and step error, in discrete-time. We conclude this chapter by proving existence and uniqueness results for the discrete-time closed loop systems considered in Chapter 4.

In Chapter 4, we derive discrete-time analogues of the continuous-time absolute stability results of [9] and [10]. Results of Popov and circle-criterion-type are presented for both a strictly causal discrete-time integrator J and an integrator J_0 with direct feedthrough. We also derive incremental versions of the circle-criterion-type results.

In Chapter 5, we apply the discrete-time absolute stability results of Chapter 4 (with strict positive real condition, that is, $\varepsilon > 0$ in (1.2) and (1.3)) to the low-gain integral control problem with input/output non-linearities and output disturbances, see Figure 5.3. In the case of constant gain and static input non-linearities it is shown that single-input single-output tracking of constant reference values is achievable, provided that, the reference value is feasible in some entirely natural sense, the steady-state gain is positive (that is, $\mathbf{G}(1) > 0$), $\mathbf{G}(z)$ is "well-behaved" as $z \to 1$ for $|z| > 1$ and the input non-linearity satisfies a local Lipschitz condition and is non-decreasing. Tracking of constant reference values is shown to persist in the presence of a large class of output disturbances, provided the integrator gain is in some interval $(0, k^*)$, where k^* is determined by the quantities

$$\sup_{q\geq 0}\left\{\operatorname{ess\,inf}_{\theta\in(0,2\pi)}\mathrm{Re}\left[\left(\frac{q}{e^{i\theta}}+\frac{1}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right]\right\}.$$

for the J integrator and

$$\sup_{q\geq 0}\left\{\operatorname{ess\,inf}_{\theta\in(0,2\pi)}\mathrm{Re}\left[\left(q+\frac{e^{i\theta}}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right]\right\}.$$

for the J_0 integrator.

In the case of time-varying gain and static input and output non-linearities, it is shown that single-input single-output tracking of constant reference values is achievable, provided that, the reference value is feasible in some entirely natural sense, $\mathbf{G}(1) > 0$, $\mathbf{G}(z)$ is "well-behaved" as $z \to 1$ for $|z| > 1$, the input/output non-linearities are non-decreasing and globally Lipschitz continuous. As before, tracking of constant reference values is shown to persist in the presence of a large class of output disturbances, provided the time-varying integrator gain is non-negative, not summable and bounded above in terms of the quantity,

$$\operatorname{ess\,inf}_{\theta\in(0,2\pi)}\mathrm{Re}\left[\frac{\mathbf{G}(e^{i\theta})}{e^{i\theta}-1}\right],$$

for the J integrator and

$$\operatorname{ess\,inf}_{\theta\in(0,2\pi)}\mathrm{Re}\left[\frac{\mathbf{G}(e^{i\theta})e^{i\theta}}{e^{i\theta}-1}\right],$$

for the J_0 integrator.

Chapter 6 is devoted to applications of the results in Chapters 4 and 5 to infinite-dimensional discrete-time state-space systems. We introduce the concept of a power stable system and a strongly stable system. We provide a non-trivial example of a strongly stable system which is not power stable. The absolute stability results from Chapter 4 with strict positive real condition (that is, $\varepsilon > 0$ in (1.2) and (1.3)) are then applied to strongly stable state-space systems and the absolute stability results with non-strict positive real condition (that is, $\varepsilon = 0$ in (1.2) and (1.3)) are applied to power stable systems. The results from Chapter 5 are applied to strongly stable state-space systems.

To enable applications of the discrete-time integral control results (see Chapter 5) to sampled-data systems, we study sample-hold discretisations of linear, shift-invariant, continuous-time systems. Previously, using an input-output approach, Helmicki, Jacobson and Nett [24, 25] studied sample-hold discretisations of distributed parameter systems belonging to the Callier-Desoer algebra. In Chapter 7, the setting is more general than in [24, 25]. At the basis of our considerations are the algebra of continuous-time shift-invariant bounded linear input-output operators on L^2 and the subalgebra consisting of all convolution operators with bounded measure kernels. We introduce the concepts of asymptotic steady-state gain, L^2-steady-state gain and step error, in continuous-time. We give results for

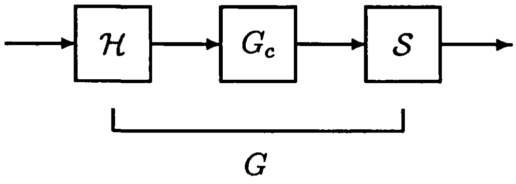

Figure 1.4: Sample-hold discretisation

continuous-time systems which, under certain natural conditions, guarantee the existence of the various steady-state gains. The main contributions of Chapter 7 are results on the behaviour of the continuous-time steady-state under sample-hold discretisations as illustrated in Figure 1.4, where G_c is a L^2-stable shift-invariant linear input-output operator, $\mathcal{H}$ is the zero-order hold operator and $\mathcal{S}$ is a generalised sampling operator. It is shown that the sample-hold discretisation of G_c, defined by $G := \mathcal{S}G_c\mathcal{H}$, is a l^2-stable shift-invariant linear input-output operator. Furthermore, we show that under a certain mild assumption on G_c, the L^2-steady-state gain of G_c and the l^2-steady-state gain of G exist and coincide. Under suitable restrictions on G_c, namely that G_c is given by convolution with a (matrix-valued) Borel measure, it is shown that this result remains true if the generalized sampling operator $\mathcal{S}$ is replaced by ideal sampling.

In Chapter 8 we apply the discrete-time theory of Chapter 5 in the context of sampled-data low-gain integral control. In the presence of a static input non-linearity both constant and time-varying gains are considered together with generalised sampling. With the additional inclusion of a static output non-linearity, we consider time-varying gain with idealised sampling, but restrict to continuous-time systems given by convolution with measure. Results are presented which guarantee that the continuous-time tracking error is asymptotically small in a certain sense, provided that, the reference value is feasible in some entirely natural sense, the steady-state gain is positive (that is, $\mathbf{G}_c(0) > 0$), $\mathbf{G}_c$ satisfies a mild assumption, and the positive-valued (possibly time-varying) integrator gain is ultimately sufficiently small and not summable. The results in Chapter 8 allow for a large class of output disturbances. We briefly discuss how through various choices of weighting function, associated with the generalised sampling operator, we can weaken some of the technical assumptions imposed in the statements of the results in Chapter 8.

Chapter 9 consists of applications of the input-output theory developed in Chapter 8 to infinite-dimensional well-posed state-space systems.

In Chapter 10, we apply results from Chapter 4 in order to study an important area of research in numerical analysis, namely, the long term behaviour of solutions of numerical methods. In particular, we discuss stability of a particular class of numerical methods, the so-called linear multistep methods. Stability analysis of linear multistep methods has previously been considered in Dahlquist [13], Nevanlinna and Odeh [49] and Nevanlinna [47, 48]. We derive stability results for linear multistep methods by applying some of the discrete-time absolute stability theory contained in Chapter 4 to a convolution equation which represents a general linear multistep method.

In Chapter 11, we briefly discuss future research topics related to this thesis.

Some technicalities have been relegated to the Appendices (Chapter 12).

Chapter 2

Notation and preliminaries

Notation

The following notation shall be used throughout the thesis.

We define $\mathbb{Z}_+ := \{x \in \mathbb{Z} \mid x \geq 0\}$, $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathbb{N} := \mathbb{Z}_+ \setminus \{0\}$. Let X be a Banach space and U be a Hilbert space. Let $F(\mathbb{Z}_+, X)$ denote the set of X-valued functions defined on $\mathbb{Z}_+$ and let $F(\mathbb{R}_+, X)$ denote the set of X-valued functions defined on $\mathbb{R}_+$. For $1 \leq p \leq \infty$, let $l^p(\mathbb{Z}_+, X)$ denote the l^p-space of unilateral X-valued sequences. In the special case $X = \mathbb{C}$, we write $l^p(\mathbb{Z}_+)$ for $l^p(\mathbb{Z}_+, \mathbb{C})$, $F(\mathbb{Z}_+)$ for $F(\mathbb{Z}_+, \mathbb{C})$ and $F(\mathbb{R}_+)$ for $F(\mathbb{R}_+, \mathbb{C})$. For $c \in \mathbb{C}$ and $R > 0$ define $\mathbb{B}(c, R) := \{z \in \mathbb{C} \mid |z - c| < R\}$. We define $\mathbb{B} := \mathbb{B}(0, 1)$. For $\alpha > 0$ define $\mathbb{E}_\alpha := \{z \in \mathbb{C} \mid |z| > \alpha\} = \mathbb{C} \setminus \overline{\mathbb{B}}(0, \alpha)$. For $\alpha \in \mathbb{R}$ define $\mathbb{C}_\alpha := \{s \in \mathbb{C} \mid \operatorname{Re} s > \alpha\}$. We denote by $\mathbb{T}$ the boundary of the set $\mathbb{E}_1$ which is also the boundary of $\mathbb{B}$. A set $S \subset X$ is called a sphere centred at $z \in X$ if there exists $\eta \geq 0$ such that $S = \{x \in X \mid \|x - z\| = \eta\}$. For a function $f : \mathbb{Z}_+ \to X$ and a subset $V \subset X$, we say that $f(n)$ approaches V as $n \to \infty$ if

$$\operatorname{dist}(f(n), V) = \inf_{v \in V} \|f(n) - v\| \to 0.$$

For a set $V \subset X$, we denote by $\operatorname{cl}(V)$ or $\overline{V}$ the closure of V in X and by $\operatorname{int}(V)$ the interior of V. We say that $V \subset X$ is *precompact* if its closure is compact. We use $\mathscr{B}(X_1, X_2)$ to denote the space of bounded linear operators from a Banach space X_1 to a Banach space X_2; we write $\mathscr{B}(X)$ for $\mathscr{B}(X, X)$. We define $\vartheta \in F(\mathbb{Z}_+)$ by $\vartheta(n) = 1$ for all $n \in \mathbb{Z}_+$ and $\vartheta_c \in F(\mathbb{R}_+)$ by $\vartheta_c(t) = 1$ for all $t \in \mathbb{R}_+$.

For $\tau \geq 0$, the *right-shift operator* $\mathrm{S}_\tau : F(\mathbb{R}_+, X) \to F(\mathbb{R}_+, X)$ is defined by,

$$(\mathrm{S}_\tau y)(t) := \begin{cases} 0, & \text{if } t \in [0, \tau), \\ y(t - \tau), & \text{if } t \geq \tau. \end{cases}$$

We say that $G : F(\mathbb{R}_+, X) \to F(\mathbb{R}_+, X)$ is (right) *shift-invariant* if $G\mathbf{S}_\tau = \mathbf{S}_\tau G$ for all $\tau \geq 0$.

Let $\mathscr{I}$ denote the integral operator given by

$$(\mathscr{I}u)(t) = \int_0^t u(s)\,ds, \qquad \forall\, u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X),\ t \in \mathbb{R}_+.$$

For any $f \in L^1(\mathbb{T}, X)$, we define the *Fourier coefficients* of f by the formula,

$$\tilde{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta}\,d\theta, \qquad n \in \mathbb{Z}.$$

We thus associate with each $f \in L^1(\mathbb{T}, X)$ a function $\tilde{f} : \mathbb{Z} \to X$. The *Fourier series* of f is

$$\sum_{n=-\infty}^{\infty} \tilde{f}(n) e^{in\theta}.$$

The subspace of all functions $w \in F(\mathbb{Z}_+, X)$ which admit a decomposition of the form $w = w_0\vartheta + w_1$ where $w_0 \in X$ and $w_1 \in l^p(\mathbb{Z}_+, X)$ for $p < \infty$ is denoted by $m^p(\mathbb{Z}_+, X) := X + l^p(\mathbb{Z}_+, X)$ (again in the special case $X = \mathbb{C}$ we denote $m^p(\mathbb{Z}_+, \mathbb{C})$ by $m^p(\mathbb{Z}_+)$). Endowed with the norm

$$\|w\|_{m^p} := \|w_0\| + \|w_1\|_{l^p},$$

the space $m^p(\mathbb{Z}_+, X)$ is complete. Let $C^\infty(\mathbb{R}_+, X)$ denote the space of infinitely differentiable functions defined on $\mathbb{R}_+$ with values in X. A function $f \in F(\mathbb{R}_+, X)$ is called piecewise continuous if there exists a sequence $0 = t_0 < t_1 < t_2 < \dots$ such that $\lim_{k\to\infty} t_k = \infty$, f is continuous on each of the intervals (t_k, t_{k+1}) and the right and left limits of f exist at each t_k. We denote the space of all piecewise continuous functions $f \in F(\mathbb{R}_+, X)$ by $PC(\mathbb{R}_+, X)$. For $\alpha \in \mathbb{R}$, we define the exponentially weighted L^p-space $L^p_\alpha(\mathbb{R}_+, X) := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}_+, X) \mid f(\cdot)\exp(-\alpha\,\cdot) \in L^p(\mathbb{R}_+, X)\}$ and endow it with the norm

$$\|f\|_{L^p_\alpha} := \left(\int_0^\infty \|e^{-\alpha t} f(t)\|^p\,dt \right)^{1/p}.$$

For $\beta > 0$, we define the weighted l^p-space $l^p_\beta(\mathbb{Z}_+, X) := \{f \in F(\mathbb{Z}_+, X) \mid f(\cdot)\beta^{-(\cdot)} \in l^p(\mathbb{Z}_+, U)\}$ and endow it with the norm

$$\|f\|_{l^p_\beta} := \left(\sum_{n=0}^{\infty} \|f(n)\beta^{-n}\|^p \right)^{1/p}.$$

We denote by $W^{1,p}(\mathbb{R}_+, X)$, where $p < \infty$, the space of all functions $f \in$

$L^p(\mathbb{R}_+, X)$ for which there exists $g \in L^p(\mathbb{R}_+, X)$ such that $f(t) - f(0) = \int_0^t g(s)\, ds$ for all $t \in \mathbb{R}_+$. We denote by $\mathscr{L}$ the Laplace transform.

We begin with several definitions of discrete-time operators that will be used throughout the thesis.

Discrete-time operators

Definition. We define the *right-shift operator* $\mathbf{S} : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ by,

$$(\mathbf{S}x)(n) := \begin{cases} 0, & \text{if } n = 0, \\ x(n-1), & \text{if } n \geq 1. \end{cases}$$

The *left-shift operator* $\mathbf{S}^* : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ is defined by,

$$(\mathbf{S}^* x)(n) := x(n+1), \qquad \forall\, n \in \mathbb{Z}_+.$$

Remark 2.1.1. If we consider $\mathbf{S} : l^2(\mathbb{Z}_+, U) \to l^2(\mathbb{Z}_+, U)$ then the left shift $\mathbf{S}^*$ is the adjoint of $\mathbf{S}$ on $l^2(\mathbb{Z}_+, U)$. $\diamond$

Definition. The *forward difference operator* $\triangle : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ is defined by

$$(\triangle x)(n) := x(n+1) - x(n), \qquad \forall\, n \in \mathbb{Z}_+.$$

The *backward difference operator* $\triangle_0 : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ is defined by

$$(\triangle_0 x)(n) := \begin{cases} x(0), & \text{if } n = 0, \\ x(n) - x(n-1), & \text{if } n \geq 1. \end{cases}$$

We now define two discrete-time integrators.

Definition. The *discrete-time integrator* $J : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ is defined by,

$$(Jy)(n) := \begin{cases} 0, & \text{if } n = 0, \\ \sum_{j=0}^{n-1} y(j), & \text{if } n \geq 1. \end{cases}$$

The *discrete-time integrator* $J_0 : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ is defined by,

$$(J_0 y)(n) := \sum_{j=0}^{n} y(j), \qquad \forall\, n \in \mathbb{Z}_+. \tag{2.1}$$

Let $x \in F(\mathbb{Z}_+, X)$. Define $E : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ by $x \mapsto x(0)\vartheta$. We now state some identities which will be used throughout the thesis.

Lemma 2.1.2.

(a) $\triangle_0 J = J\triangle_0 = \mathbf{S}$ *and* $\triangle_0 J_0 = J_0\triangle_0 = I$.

(b) $J\triangle = I - E$ *and* $\triangle J = I$.

(c) $J_0\triangle = \mathbf{S}^* - E$ *and* $\triangle J_0 = \mathbf{S}^*$.

The proof of Lemma 2.1.2 is routine and therefore omitted.

Shift-invariance and causality

Definition. We say that $H : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ is (right) *shift-invariant* if $H\mathbf{S} = \mathbf{S}H$. Throughout the thesis 'shift-invariant' means invariant with respect to right shifts.

We aim to show that every linear, shift-invariant operator $H : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ is causal.

Definition. Let $H : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$. Then we define H to be *weakly causal* if, $x(n) = 0$ for $n \leq N$ implies that $(Hx)(n) = 0$ for $n \leq N$, for all $N \in \mathbb{Z}_+$, $x \in F(\mathbb{Z}_+, X)$.

Definition. Let $\mathbf{P}_N : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ be the linear map defined by,

$$(\mathbf{P}_N f)(n) := \begin{cases} f(n), & n \leq N, \\ 0, & n > N. \end{cases}$$

if nticipative) if and only

$$\mathbf{P}_N H \mathbf{P}_N = \mathbf{P}_N H, \quad \forall\, N \in \mathbb{Z}_+.$$

Equivalently, H is said to be causal if, for all $x, y \in F(\mathbb{Z}_+, X)$

Equivalently, H is said to be causal if, for all $x, y \in F(\mathbb{Z}_+, X)$,

$$x(n) = y(n), \quad n \leq N \Rightarrow (Hx)(n) = (Hy)(n), \quad n \leq N, \quad \forall\, N \in \mathbb{Z}_+.$$

Remark 2.1.3. We say that $H : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ is *strictly causal* if, for all $x, y \in F(\mathbb{Z}_+, X)$,

$$x(n) = y(n), \quad n < N \Rightarrow (Hx)(n) = (Hy)(n), \quad n \leq N, \quad \forall\, N \in \mathbb{N}.$$

Note that, in particular, the discrete-time integrator J is strictly causal, whereas J_0 is not. ◇

The following three results are an adaptation of arguments in [61].

Proposition 2.1.4. *If $H : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ is linear and weakly causal then it is causal.*

Proof. Suppose that $x(n) = y(n)$ for $n \leq N$, where $N \in \mathbb{Z}_+$. Define $z(n) := x(n) - y(n) = 0$ for $n \leq N$. Then by weak causality we have $(Hz)(n) = 0$ for $n \leq N$ and so $(H(x-y))(n) = 0$ for $n \leq N$ by definition of z. Hence by linearity $(Hx)(n) - (Hy)(n) = 0$ for $n \leq N$ and so we see that $(Hx)(n) = (Hy)(n)$ for $n \leq N$ and H is causal. □

Proposition 2.1.5. *If $H : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ is shift-invariant, then it is weakly causal.*

Proof. Suppose $N \in \mathbb{Z}_+$, $x \in F(\mathbb{Z}_+, X)$ and that $x(n) = 0$ for all $n \leq N$. Define, $y(n) = x(n + N)$ for all $n \in \mathbb{Z}_+$. Then it is clear by definition that $y \in F(\mathbb{Z}_+, X)$ and that, $x = \mathbf{S}^{N+1}y$. Since H is shift-invariant, it follows that,

$$Hx = H\mathbf{S}^{N+1}y = \mathbf{S}^{N+1}Hy.$$

By definition of $\mathbf{S}$ we see that, $(Hx)(n) = (\mathbf{S}^{N+1}Hy)(n) = 0$ whenever $n \leq N$, so that H is weakly causal. □

Combining Proposition 2.1.4 and 2.1.5, we have the following corollary.

Corollary 2.1.6. *If $H : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ is linear and shift-invariant, then it is causal.*

A result on ω-limit sets

Definition. For $x \in F(\mathbb{Z}_+, X)$, the *ω-limit set Ω of x* is defined to be

$$\Omega := \{\xi \in X \mid x(n_j) \to \xi \text{ for some sequence } n_j \to \infty\}.$$

We shall require the following result.

Lemma 2.1.7. *Let $x \in F(\mathbb{Z}_+, X)$ and suppose that $\{x(n) \mid n \in \mathbb{Z}_+\}$ is precompact. If $(\triangle_0 x)(n) \to 0$ as $n \to \infty$ then, the ω-limit set Ω of x is connected.*

Proof. First note that since $\{x(n) \mid n \in \mathbb{Z}_+\}$ is precompact, x has a convergent subsequence. Consequently, we deduce that Ω is non-empty. It remains to show that Ω is connected. Suppose that Ω is not connected. Then Ω is the union of two disjoint closed non-empty sets Ω_1 and Ω_2. Since Ω is compact, so are Ω_1 and Ω_2. Hence, there exist open neighbourhoods U_1, U_2 of Ω_1, Ω_2 in X, such that $U_1 \cap U_2 = \emptyset$ and

$$\operatorname{dist}(U_1, U_2) > \varepsilon, \tag{2.2}$$

for some $\varepsilon > 0$. Furthermore, there exists sequences n_j, n'_j with $n_j \to \infty$, $n'_j \to \infty$ as $j \to \infty$ such that

$$\begin{aligned} x(n_j) &\to \xi_1 \in \Omega_1, \qquad & j \to \infty, \\ x(n'_j) &\to \xi_2 \in \Omega_2, \qquad & j \to \infty. \end{aligned}$$

Hence for N, N' sufficiently large, $x(n_{N+j}) \in U_1$ and $x(n'_{N'+j}) \in U_2$ for all j. Now consider the sequence m_j defined by

$$m_j := \begin{cases} n_j, & \text{if } j \text{ is even}, \\ n'_j, & \text{if } j \text{ is odd}. \end{cases}$$

By definition of m_j, for $M = \max\{N, N'\}$, the sequence $x(m_{M+j})$ jumps alternately between the open sets U_1 and U_2. However, by assumption, $(\Delta_0 x)(m_j) \to 0$ as $j \to \infty$, contradicting (2.2). □

Measure theory

The *Borel σ-algebra on* $\mathbb{R}_+$ is the σ-algebra generated by the family of open sets in $\mathbb{R}_+$ and is denoted by $\mathcal{B}_{\mathbb{R}_+}$. Its members are called *Borel sets.*

Definition. Let $m, n \in \mathbb{N}$. We say that $\mu : \mathcal{B}_{\mathbb{R}_+} \to \mathbb{C}^{m\times n}$ is a $\mathbb{C}^{m\times n}$-valued *Borel measure* on $\mathbb{R}_+$, if,

(i) $\mu(\emptyset) = 0_{m\times n}$,

(ii) for any sequence $\{E_j\}$ of disjoint sets in $\mathcal{B}_{\mathbb{R}_+}$,

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j), \tag{2.3}$$

observe that the convergence of the series in (2.3) is part of the requirement.

If, in the above definition, $m = n = 1$ and $\mu(E) \in \mathbb{R}$ for all $E \in \mathcal{B}_{\mathbb{R}_+}$, then μ is also called a finite signed Borel measure. If, in the above definition, $m = n = 1$ and $\mu(E) \in \mathbb{R}_+$ for all $E \in \mathcal{B}_{\mathbb{R}_+}$, then μ is also called a finite non-negative Borel measure.

Let E be a set. We call $\{E_j\}_{j=1}^{N}$ a *partition* of E if $\cup_{j=1}^{N} E_j = E$ and $E_i \cap E_j = \emptyset$ for all $i \neq j$.

Definition. Let μ be a $\mathbb{C}^{m\times n}$-valued Borel measure on $\mathbb{R}_+$. The *total variation*

$|\mu|(E)$ of μ on a set $E \in \mathcal{B}_{\mathbb{R}_+}$ is given by

$$|\mu|(E) = \sup \sum_{j=1}^{N} \|\mu(E_j)\|$$

where the supremum is taken over all $N \in \mathbb{Z}_+$ and over all partitions $\{E_j\}$ of E with $E_j \in \mathcal{B}_{\mathbb{R}_+}$.

Theorem 2.1.8. *Let μ be a $\mathbb{C}^{m\times n}$-valued Borel measure on $\mathbb{R}_+$. Then the total variation $|\mu|$ of μ is a finite non-negative Borel measure on $\mathbb{R}_+$.*

Proof. The proof of the non-negativity of $|\mu|$ is an exact copy of the proof of the scalar case, see, for example, [51], Theorem 6.2. It remains to show that $|\mu|$ is finite. Define $\mu_{ij} : \mathcal{B}_{\mathbb{R}_+} \to \mathbb{C}$ by

$$\mu_{ij}(E) := (\mu(E))_{ij}, \qquad 1 \le i \le m,\ 1 \le j \le n,\ E \in \mathcal{B}_{\mathbb{R}_+}.$$

Then μ_{ij} defines a complex Borel measure for $1 \le i \le m$, $1 \le j \le n$. Hence it follows from [51], Theorem 6.4, that

$$|\mu_{ij}|(\mathbb{R}_+) < \infty, \qquad 1 \le i \le m,\ 1 \le j \le n. \tag{2.4}$$

Let $\{E_k\}_{k=1}^{N}$ be a partition of $\mathbb{R}_+$. Now

$$\sum_{k=1}^{N} \|\mu(E_k)\| \le \alpha \sum_{k=1}^{N} \sum_{i,j} |\mu_{ij}(E_k)| = \alpha \sum_{i,j} \sum_{k=1}^{N} |\mu_{ij}(E_k)| \le \alpha \sum_{i,j} |\mu_{ij}|(\mathbb{R}_+)$$
$$\le \alpha nm |\mu_{ij}|(\mathbb{R}_+), \tag{2.5}$$

where α depends on the choice of norm in $\mathbb{C}^{m\times n}$. Defining $\gamma := \alpha nm|\mu_{ij}|(\mathbb{R}_+)$ we see from (2.4) that $\gamma < \infty$. Hence by (2.5),

$$|\mu|(\mathbb{R}_+) = \sup \sum_{k=1}^{N} \|\mu(E_k)\| \le \gamma,$$

showing that $|\mu|$ is finite. $\square$

Corollary 2.1.9. *Let μ be a $\mathbb{C}^{m\times n}$-valued Borel measure on $\mathbb{R}_+$. Then*

$$\|\mu(E)\| \le |\mu|(\mathbb{R}_+) < \infty, \qquad \forall\, E \in \mathcal{B}_{\mathbb{R}_+},$$

that is, μ is bounded.

Proof. Since $|\mu|$ is a non-negative measure, $|\mu|(E) \le |\mu|(\mathbb{R}_+)$ for all $E \in \mathcal{B}_{\mathbb{R}_+}$

and so

$$\|\mu(E)\| \leq |\mu|(E) \leq |\mu|(\mathbb{R}_+) < \infty, \qquad \forall\, E \in \mathcal{B}_{\mathbb{R}_+}.$$

□

The following result will be required in Chapter 7.

Theorem 2.1.10. *Let μ be a $\mathbb{C}^{m\times n}$-valued Borel measure on $\mathbb{R}_+$. Let $p \in \mathbb{N}$ and φ be a $\mathbb{C}^{n\times p}$-valued $|\mu|$-integrable function on $\mathbb{R}_+$. Then*

$$\left\| \int_0^\infty \mu(ds)\varphi(s) \right\| \leq \int_0^\infty \|\varphi(s)\| |\mu|(ds).$$

A proof of Theorem 2.1.10 can be found in [21], see Chapter 3, Theorem 5.6, of [21].

Definition. The convolution $\mu * v$ of a $\mathbb{C}^{m\times n}$-valued Borel measure μ on $\mathbb{R}_+$ and $v \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^n)$ is the function

$$(\mu * v)(t) = \int_0^t \mu(ds)v(t-s),$$

defined for those t for which the function $s \mapsto v(t-s)$ is locally $|\mu|$-integrable. Note that it follows from [20] (see, [20], Proposition 2.12) that if v is Lebesgue measurable, then without loss of generality, we may assume that v is Borel measurable in the following sense: there exists Borel measurable $\widetilde{v}$ such that $v = \widetilde{v}$ a.e. in the Lebesgue sense. Moreover, by Proposition 8.49 of [20], it follows that $s \mapsto v(t-s)$ is locally $|\mu|$-integrable for a.e. t (here a.e. refers to the Lebesgue measure).

Limits of functions in $L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$

Each element of $L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ is an equivalence class of locally integrable functions that coincide almost everywhere on $\mathbb{R}_+$. For later purposes we need to make sense of the limit of 'functions' in $L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$. Let $f : \mathbb{R}_+ \to X$ be a locally integrable function and let $[f] \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ be the corresponding equivalence class of all measurable functions in $F(\mathbb{R}_+, X)$ which coincide with f almost everywhere.

Definition. We say that $[f] \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ has a limit $l \in X$ as $t \to \infty$ if, there exists a representative $g : \mathbb{R}_+ \to X$ of $[f]$ such that $\lim_{t\to\infty} g(t) = l$ (in the usual sense) and we write $\lim_{t\to\infty}[f](t) = l$.

Remark 2.1.11. Note that this definition is independent of the choice of representative from the equivalence class. If g_1, g_2 were two representatives of $[f] \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$, such that, $\lim_{t\to\infty} g_1(t) \neq \lim_{t\to\infty} g_2(t)$, then, g_1 and g_2 would differ on a set of positive measure contradicting the fact that g_1 and g_2 are representatives of the same equivalence class. $\diamond$

Definition. We say that a measurable function $f : \mathbb{R}_+ \to X$ has an *essential limit* at infinity if there exists $l \in X$ such that $\operatorname{ess\,sup}_{T \geq t} \|f(T) - l\|$ tends to 0 as $t \to \infty$ and we write $\operatorname{ess\,lim}_{t\to\infty} f(t) = l$.

Proposition 2.1.12. *Let $f : \mathbb{R}_+ \to X$ be a measurable function such that $[f] \in L^1_{\text{loc}}(\mathbb{R}_+, X)$. Then $\lim_{t\to\infty}[f](t) = l$ if and only if $\operatorname{ess\,lim}_{t\to\infty} f(t) = l$.*

Proof. ($\Rightarrow$) Suppose that $\lim_{t\to\infty}[f](t) = l$. Then, there exists a representative $g : \mathbb{R}_+ \to X$ of $[f]$ such that $\lim_{t\to\infty} g(t) = l$. Hence, for all $\varepsilon > 0$, there exists $t_0 \geq 0$ such that, whenever $t \geq t_0$,

$$\|g(t) - l\| < \varepsilon.$$

Consequently,

$$\sup_{t \geq t_0} \|g(t) - l\| \leq \varepsilon,$$

showing that

$$\lim_{t_0 \to \infty} \left(\sup_{t \geq t_0} \|g(t) - l\| \right) = 0.$$

Hence, $\operatorname{ess\,lim}_{t\to\infty} f(t) = l$.

($\Leftarrow$) Suppose that $\operatorname{ess\,lim}_{t\to\infty} f(t) = l$. Then,

$$\lim_{t\to\infty} \operatorname{ess\,sup}_{T \geq t} \|f(T) - l\| = 0.$$

Hence, for all $\varepsilon > 0$, there exists $t_0 \geq 0$ such that, whenever $t \geq t_0$

$$\operatorname{ess\,sup}_{T \geq t} \|f(T) - l\| < \varepsilon.$$

That is,

$$\|f(T) - l\| < \varepsilon, \qquad \text{a.e. } T \geq t \geq t_0.$$

Consequently, there exists a sequence $t_n \uparrow \infty$ $(n \in \mathbb{N})$ such that

$$\|f(t) - l\| \leq 1/n, \qquad \text{a.e. } t \geq t_n.$$

Define $g : \mathbb{R}_+ \to X$ by

$$g(t) := \begin{cases} f(t), & t \in [0, t_1), \\ f(t), & t \in [t_n, t_{n+1}) \text{ and } \|f(t) - l\| \leq 1/n, \\ l, & t \in [t_n, t_{n+1}) \text{ and } \|f(t) - l\| > 1/n. \end{cases}$$

Then $g(t) = f(t)$ a.e. and $\lim_{t\to\infty} g(t) = l$. Hence $\lim_{t\to\infty}[f](t) = l$. $\square$

Throughout the rest of the thesis we shall not use the notation $[f]$: we simply write $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ with the understanding that f is the equivalence class of all measurable functions which coincide with f almost everywhere.

In later chapters we require the following results.

Proposition 2.1.13. *Let $f \in L^p(\mathbb{R}_+, X)$ for some $p \in [1, \infty)$. Then for all $\varepsilon > 0$, the Lebesgue measure of the set $E := \{t \geq T \mid \|f(t)\| \geq \varepsilon\}$ tends to 0 as $T \to \infty$.*

Proof. Let m denote the Lebesgue measure. Since $f \in L^p(\mathbb{R}_+, X)$ for some $p \in [1, \infty)$ we have

$$\gamma(T) := \int_T^\infty \|f(t)\|^p \, dt \geq \int_E \|f(t)\|^p \, dt \geq \varepsilon^p m(E), \quad \forall\, \varepsilon > 0.$$

Consequently, $m(E) \leq \gamma(T)\varepsilon^{-p}$ for all $\varepsilon > 0$. Since $\lim_{T\to\infty} \gamma(T) = 0$ it follows that $m(E) \to 0$ as $T \to \infty$. $\square$

Proposition 2.1.14. *Let $f, \dot{f} \in L^2(\mathbb{R}_+, U)$ where U denotes a Hilbert space. Then $\lim_{t\to\infty} f(t) = 0$.*

Proof. Note that,

$$\frac{d}{dt}(\|f(t)\|^2) = \langle \dot{f}(t), f(t)\rangle + \langle f(t), \dot{f}(t)\rangle = 2\mathrm{Re}\,\langle f(t), \dot{f}(t)\rangle.$$

Hence it follows that

$$\|f(t)\|^2 - \|f(0)\|^2 = 2\mathrm{Re} \int_0^t \langle f(\tau), \dot{f}(\tau)\rangle \, d\tau. \tag{2.6}$$

Since $f, \dot{f} \in L^2(\mathbb{R}_+, U)$ we have

$$2\mathrm{Re} \int_0^t \langle f(\tau), \dot{f}(\tau)\rangle \, d\tau \to l, \qquad t \to \infty.$$

Consequently, by (2.6) we see that $\lim_{t\to\infty} \|f(t)\| =: f^\infty$ exists. Noting further that $f \in L^2(\mathbb{R}_+, U)$, we deduce that $f^\infty = 0$. $\square$

Chapter 3

Convolutions of sequences, $\mathscr{Z}$-transforms and transfer functions of discrete-time operators

In this chapter we discuss three key notions; the convolution of two sequences, the $\mathscr{Z}$-transform of a sequence, and transfer functions of bounded linear shift-invariant operators on $l^2(\mathbb{Z}_+, U)$. We introduce the concepts of asymptotic steady-state gain, l^2-steady-state gain and step error in discrete-time. This is done in the general context of the algebra of all shift-invariant bounded linear input-output operators on $l^2(\mathbb{Z}_+, U)$. We conclude this chapter by deriving existence and uniqueness results for the discrete-time equations which are to be considered in later chapters.

3.1 Convolutions of sequences

Definition. Suppose $x \in F(\mathbb{Z}_+, \mathscr{B}(X))$ and $y \in F(\mathbb{Z}_+, X)$. Then their convolution $x * y \in F(\mathbb{Z}_+, X)$ is defined by,

$$(x * y)(n) = \sum_{j=0}^{n} x(n-j)y(j) = \sum_{j=0}^{n} x(j)y(n-j).$$

The convolution product in the space $F(\mathbb{Z}_+)$ is commutative and the sequence δ defined by

$$\delta(n) := \begin{cases} 1, & n = 0, \\ 0, & n \in \mathbb{N}, \end{cases}$$

is the unit element. A sequence $a \in F(\mathbb{Z}_+)$ is invertible (that is, there exists $a^{-1} \in F(\mathbb{Z}_+)$ such that $a * a^{-1} = a^{-1} * a = \delta$) if and only if $a(0) \neq 0$.

Note that $J_0 x$ of $x \in F(\mathbb{Z}_+, X)$ can be represented by convolution with ϑ:

$$(J_0 x)(n) := \sum_{j=0}^{n} x(j) = (\vartheta * x)(n).$$

The following result is a standard result on convolution of sequences.

Lemma 3.1.1. *Assume that $a \in l^1(\mathbb{Z}_+)$ and let $1 \leq p \leq \infty$. The following statements hold:*

(a) $\|a * b\|_{l^p} \leq \|a\|_{l^1} \|b\|_{l^p}$ *for all* $b \in l^p(\mathbb{Z}_+, X)$.

(b) *If* $b \in F(\mathbb{Z}_+, X)$, *then*

$$\lim_{n \to \infty} b(n) = 0 \quad \Longrightarrow \quad \lim_{n \to \infty} (a * b)(n) = 0.$$

Proof. We refer to [15], p. 244, for the proof of part (a). To prove part (b), assume that $b(n) \to 0$ as $n \to \infty$. For every $n \in \mathbb{Z}_+$, let m_n denote the largest integer less than or equal to $n/2$. Since

$$(a * b)(n) = \sum_{k=0}^{m_n} a(n-k) b(k) + \sum_{k=m_n+1}^{n} a(n-k) b(k),$$

we obtain

$$\|(a * b)(n)\| \leq \|b\|_{l^\infty} \sum_{k \geq n/2} |a(k)| + \|a\|_{l^1} \sup_{k \geq n/2} \|b(k)\|.$$

By assumption $b(k) \to 0$ as $k \to \infty$ and since $a \in l^1$,

$$\lim_{n \to \infty} \sum_{k \geq n/2} |a(k)| = 0.$$

Consequently, $(a * b)(n) \to 0$ as $n \to \infty$. □

We now show that every linear shift-invariant operator on $F(\mathbb{Z}_+, X)$ is a convolution operator.

Proposition 3.1.2. *Let $H : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ be linear and shift-invariant. Then,*

$$(Hy)(n) = \sum_{k=0}^{n} h(n-k) y(k), \qquad \forall\, y \in F(\mathbb{Z}_+, X),\ n \in \mathbb{Z}_+,$$

where, for each $n \in \mathbb{Z}_+$, $h(n) : X \to X$ is defined by

$$h(n)\xi := (H(\xi\delta))(n), \quad \xi \in X.$$

Proof. Let $y \in F(\mathbb{Z}_+, X)$. By causality of H,

$$(Hy)(n) = \left(H\left(\sum_{k=0}^{n} y(k)\mathrm{S}^k\delta\right)\right)(n), \qquad \forall\, n \in \mathbb{Z}_+.$$

Using first the linearity of H, then the shift-invariance of H, and finally the definition of h, we see that,

$$\begin{aligned}(Hy)(n) = \left(H\left(\sum_{k=0}^{n} y(k)\mathrm{S}^k\delta\right)\right)(n) &= \sum_{k=0}^{n}(Hy(k)\mathrm{S}^k\delta)(n) \\ &= \sum_{k=0}^{n}(\mathrm{S}^k H(y(k)\delta))(n) \\ &= \sum_{k=0}^{n}(Hy(k)\delta)(n-k) \\ &= \sum_{k=0}^{n} h(n-k)y(k), \qquad \forall\, n \in \mathbb{Z}_+.\end{aligned}$$

□

Remark 3.1.3. Note that if $\dim X < \infty$ and $H \in \mathscr{B}(l^2(\mathbb{Z}_+, X))$ is shift-invariant, then the convolution kernel h of H, as defined in Proposition 3.1.2, is in $l^2(\mathbb{Z}_+, \mathscr{B}(X))$. To see that this is not the case if X is infinite-dimensional, we consider the following example. Let $X = l^2(\mathbb{Z}_+, \mathbb{R})$ and $\{e^j\}_{j\in\mathbb{Z}_+}$ be the standard orthonormal basis of X. Let $v \in X$. Then,

$$v = \sum_{j=0}^{\infty} v_j e^j.$$

Define $k(j) \in \mathscr{B}(X)$ by

$$k(j)v := v_j e^j, \qquad v \in X.$$

Then, for each $j \in \mathbb{Z}_+$, $\|k(j)\|_X = 1$, so that $k \notin l^2(\mathbb{Z}_+, \mathscr{B}(X))$. Define $T : F(\mathbb{Z}_+, X) \to F(\mathbb{Z}_+, X)$ by

$$(Tu)(n) := \sum_{j=0}^{n} k(n-j)u(j), \qquad \forall\, u \in F(\mathbb{Z}_+, X).$$

Let $u \in l^2(\mathbb{Z}_+, X)$. Then,

$$\sum_{n=0}^{\infty} \|(Tu)(n)\|_X^2 = \sum_{n=0}^{\infty} \left\| \sum_{j=0}^{n} k(n-j)u(j) \right\|_X^2 = \sum_{n=0}^{\infty} \left\| \sum_{j=0}^{n} u_{n-j}(j)e^j \right\|_X^2. \tag{3.1}$$

Since the $\{e^j\}_{j\in\mathbb{Z}_+}$ form an orthonormal basis of X, we have

$$\left\| \sum_{j=0}^{n} u_{n-j}(j)e^j \right\|_X^2 = \sum_{j=0}^{n} (u_{n-j}(j))^2. \tag{3.2}$$

Combining (3.1) and (3.2) we obtain,

$$\sum_{n=0}^{\infty} \|(Tu)(n)\|_X^2 = \sum_{n=0}^{\infty} \sum_{j=0}^{n} (u_{n-j}(j))^2.$$

Setting $u_{n-j}(j) := 0$ if $j > n$, we have,

$$\begin{aligned}
\sum_{n=0}^{\infty} \|(Tu)(n)\|_X^2 = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (u_{n-j}(j))^2 &= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} (u_{n-j}(j))^2 \\
&= \sum_{n=0}^{\infty} (u_n(0))^2 + \sum_{n=0}^{\infty} (u_{n-1}(1))^2 + \dots \\
&= \|u(0)\|_X^2 + \|u(1)\|_X^2 + \dots \\
&= \sum_{n=0}^{\infty} \|u(n)\|_X^2 \\
&= \|u\|_{l^2(\mathbb{Z}_+,X)}^2.
\end{aligned} \tag{3.3}$$

Hence we see from (3.3) that, $Tu \in l^2(\mathbb{Z}_+, X)$ and $\|Tu\|_{l^2(\mathbb{Z}_+,X)} = \|u\|_{l^2(\mathbb{Z}_+,X)}$, so that $T \in \mathscr{B}(l^2(\mathbb{Z}_+, X))$. ◇

Corollary 3.1.4. *Let G, $H : F(\mathbb{Z}_+) \to F(\mathbb{Z}_+)$ be linear shift-invariant operators. Then, $GH = HG$.*

Proof. Let $v \in F(\mathbb{Z}_+)$. Since G, $H : F(\mathbb{Z}_+) \to F(\mathbb{Z}_+)$ are linear and shift-invariant it follows from Proposition 3.1.2 that there exists $h, g \in F(\mathbb{Z}_+)$ such that

$$(Hv)(n) = (h * v)(n), \qquad \forall\, v \in F(\mathbb{Z}_+),$$

and

$$(Gv)(n) = (g * v)(n), \qquad \forall\, v \in F(\mathbb{Z}_+).$$

Noting that the convolution product in $F(\mathbb{Z}_+)$ is commutative it follows that,

$$(HGv)(n) = (h * g * v)(n) = (g * h * v)(n) = (GHv)(n).$$

$\square$

3.2 The $\mathscr{Z}$-transform

Definition. The $\mathscr{Z}$ transform of $a \in F(\mathbb{Z}_+, X)$ is defined by,

$$\hat{a}(z) = (\mathscr{Z}(a))(z) = \sum_{j=0}^{\infty} a(j) z^{-j}, \tag{3.4}$$

where z is a complex variable. We say that a is *$\mathscr{Z}$-transformable* if the series in (3.4) converges for some $z = z_0 \in \mathbb{C} \setminus \{0\}$, in which case it converges absolutely for all $z \in \mathbb{C}$ with $|z| > |z_0|$.

Remark 3.2.1. It is an elementary fact from the theory of power series, (see e.g. [2] p.31), that a is $\mathscr{Z}$-transformable if and only if

$$r_a := \limsup_{n \to \infty} \|a(n)\|^{1/n} < \infty, \tag{3.5}$$

in which case (3.4) converges absolutely if $|z| > r_a$ and diverges if $|z| < r_a$. $\diamond$

For $\eta > r_a$ we have that

$$\hat{a}(\eta e^{i\theta}) = \sum_{k=0}^{\infty} (a(k)\eta^{-k}) e^{-ik\theta}, \quad \theta \in [0, 2\pi),$$

showing that the function $\theta \mapsto \hat{a}(\eta e^{i\theta})$ is the discrete Fourier transform of the sequence $(a(k)\eta^{-k})_{k \in \mathbb{Z}_+}$. If $a \in F(\mathbb{Z}_+, X)$ is $\mathscr{Z}$-transformable, then, for every $\eta > r_a$, the function $\hat{a}$ is holomorphic and bounded on $\mathbb{E}_\eta$. Conversely if $\eta > 0$ and $A : \mathbb{E}_\eta \to X$ is holomorphic and bounded, then $a \in F(\mathbb{Z}_+, X)$ defined by

$$a(n) := \frac{1}{2\pi i} \int_{|z| = \nu} A(z) z^{n-1}\, dz = \frac{\nu^n}{2\pi} \int_0^{2\pi} A(\nu e^{i\theta}) e^{in\theta}\, d\theta, \quad \text{where } \nu > \eta,$$

is the unique $\mathscr{Z}$-transformable sequence (with $r_a \le \eta$) such that $\hat{a}(z) = A(z)$ for all $z \in \mathbb{E}_\eta$ and we write $a = \mathscr{Z}^{-1}(A)$.

Under the $\mathscr{Z}$-transform, convolutions become multiplications, that is, if $a \in F(\mathbb{Z}_+, \mathscr{B}(X))$ and $b \in F(\mathbb{Z}_+, X)$ are $\mathscr{Z}$-transformable, then $a * b$ is $\mathscr{Z}$-transformable

and

$$\widehat{(a*b)}(z) = \hat{a}(z)\hat{b}(z), \qquad z \in \mathbb{C} \text{ s.t. } |z| > \max\{r_a,\, r_b\}.$$

3.3 Transfer functions of discrete-time operators

In order to discuss transfer functions of bounded linear shift-invariant operators on $l^2(\mathbb{Z}_+, U)$, we first need to define the Hardy-Lebesgue spaces H^2 and H^∞. To this end, we have the following result; a proof of which can be found in [51] (see, Theorem 17.6 in [51]).

Theorem 3.3.1. *If $f : \mathbb{C} \to X$ is holomorphic on $\mathbb{E}_1 \cup \{\infty\}$, and if*

$$\begin{aligned} M_2(f;r) &:= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \|f(re^{i\theta})\|^2\, d\theta\right)^{1/2}, \\ M_\infty(f;r) &:= \sup_{\theta\in[0,2\pi)} \|f(re^{i\theta})\|, \end{aligned}$$

then M_2 and M_∞ are monotonically decreasing functions of r for $r > 1$.

Definition. Let $f : \mathbb{C} \to X$ be holomorphic on $\mathbb{E}_1 \cup \{\infty\}$. Setting,

$$\|f\|_{H^2} := \lim_{r\downarrow 1} M_2(f;r), \qquad \|f\|_{H^\infty} := \lim_{r\downarrow 1} M_\infty(f;r),$$

the Hardy-Lebesgue spaces $H^2(\mathbb{E}_1, X)$ and $H^\infty(\mathbb{E}_1, X)$ are defined to consist of all f for which $\|f\|_{H^2} < \infty$ and $\|f\|_{H^\infty} < \infty$, respectively. Note that if $f \in H^\infty(\mathbb{E}_1, X)$, then $\|f\|_{H^\infty} = \sup_{z\in\mathbb{E}_1} \|f(z)\|$. Moreover, $H^\infty(\mathbb{E}_1, X) \subseteq H^2(\mathbb{E}_1, X)$ and $\|f\|_{H^2} \leq \|f\|_{H^\infty}$ for every $f \in H^\infty(\mathbb{E}_1, X)$.

The basic properties of $H^2(\mathbb{E}_1, X)$ are stated in the following theorem.

Theorem 3.3.2. (a) *A function $f : \mathbb{E}_1 \to X$ is in $H^2(\mathbb{E}_1, X)$ if and only if there exists $x \in l^2(\mathbb{Z}_+, X)$ such that $\hat{x} = f$. Moreover, $\|f\|_{H^2} = \|x\|_{l^2}$.*

(b) *If $f \in H^2(\mathbb{E}_1, X)$, then f has radial limits $f^*(e^{i\theta})$ at almost all points of $\mathbb{T}$ and $f^* \in L^2(\mathbb{T}, X)$. The nth Fourier coefficient of f^* is $x(-n)$ if $n \leq 0$ and 0 if $n > 0$, where $x \in l^2(\mathbb{Z}_+, X)$ is such that $\hat{x} = f$.*

(c) *The mapping $f \to f^*$ is an isometry of $H^2(\mathbb{E}_1, X)$ onto the subspace of $L^2(\mathbb{T}, X)$ which consists of those $g \in L^2(\mathbb{T}, X)$ which have $\tilde{g}(n) = 0$ for all $n > 0$ (where $\tilde{g}(n)$ denote the Fourier coefficients of g).*

Proof. Using the fact that $h \in H^2(\mathbb{E}_1, X)$ if and only if the function $g(z) := h(1/z)$ is in $H^2(\mathbb{B}, X)$, for all $z \in \mathbb{B}$, the proof of the scalar case, see, for example, Theorem 17.10 of [51], carries over word for word to obtain the claim. $\square$

Now let $u, v \in l^2(\mathbb{Z}_+, U)$ with associated $\mathscr{Z}$-transforms $\hat{u}, \hat{v}$. By Theorem 3.3.2 we know that $\hat{u}, \hat{v} \in H^2(\mathbb{E}_1, U)$ and that the radial limits,

$$\begin{aligned} \hat{u}(e^{i\theta}) &:= \lim_{r\downarrow 1} \hat{u}(re^{i\theta}) \\ \hat{v}(e^{i\theta}) &:= \lim_{r\downarrow 1} \hat{v}(re^{i\theta}) \end{aligned}$$

exist for almost all $\theta \in [0, 2\pi)$. We now state a theorem we shall require in Chapter 4.

Theorem 3.3.3 (Parseval-Bessel). *For $u, v \in l^2(\mathbb{Z}_+, U)$ we have that*

$$\sum_{n=0}^{\infty} \langle u(n), v(n) \rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle \hat{u}(e^{i\theta}), \hat{v}(e^{i\theta}) \rangle \, d\theta.$$

Proof. A proof of the Parseval-Bessel Theorem can be found in [51], page 92. The above statement is slightly different, but follows from the version in [51] combined with Theorem 3.3.2 (a). $\square$

Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ be shift-invariant. Then G has a transfer function $\mathbf{G} \in H^\infty(\mathbb{E}_1, \mathscr{B}(U))$ in the sense that,

$$(\mathscr{Z}(Gu))(z) = \mathbf{G}(z)(\mathscr{Z}(u))(z), \qquad \forall\, u \in l^2(\mathbb{Z}_+, U),\ z \in \mathbb{E}_1.$$

We see this from the following theorem.

Theorem 3.3.4. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ be shift-invariant. Then there is a unique $\mathbf{G} \in H^\infty(\mathbb{E}_1, \mathscr{B}(U))$ such that, for any $u \in l^2(\mathbb{Z}_+, U)$, denoting $y = Gu$,*

$$\hat{y}(z) = \mathbf{G}(z)\hat{u}(z), \qquad \forall\, z \in \mathbb{E}_1. \tag{3.6}$$

Moreover, $\|G\| = \|\mathbf{G}\|_{H^\infty}$.

Remark 3.3.5. A proof of Theorem 3.3.4 can be found in [50] (see [50], Theorem B, p.15). The statement of the result in [50] is rather different to the statement of Theorem 3.3.4 and requires the following translations. In the notation of [50], S, T and A are replaced by $\mathbf{S}$, G and $\mathbf{G}$, respectively. In [50], S is given by multiplication by z on $H^2(\mathbb{B}, U)$ and corresponds to $\mathbf{S}$ via the isometric isomorphism between $H^2(\mathbb{B}, U)$ and $l^2(\mathbb{Z}_+, U)$. The bounded linear operator $T : H^2(\mathbb{B}, U) \to H^2(\mathbb{B}, U)$ is assumed to be S-analytic, that is, S and T commute. Again via the isometric isomorphism between $H^2(\mathbb{B}, U)$ and $l^2(\mathbb{Z}_+, U)$ this is equivalent in our terms to $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ being shift-invariant. Finally, noting that $h \in H^\infty(\mathbb{E}_1, \mathscr{B}(U))$ if and only if the function $g : z \mapsto h(1/z)$ is in $H^\infty(\mathbb{B}, \mathscr{B}(U))$, it follows that A in [50] represents the transfer function $\mathbf{G}$ as defined in the statement of Theorem 3.3.4. $\diamond$

In the following set of examples we compute the transfer functions of the shift-invariant operators $\mathbf{S}, \Delta_0, J$ and J_0.

Examples 3.3.6. (a) Let $x \in F(\mathbb{Z}_+, U)$ and $z \in \mathbb{E}_1$. We compute the $\mathscr{Z}$-transform of $\mathbf{S}x$ as follows

$$(\widehat{\mathbf{S}x})(z) := \sum_{j=0}^{\infty} (\mathbf{S}x)(j) z^{-j} = \sum_{j=1}^{\infty} x(j-1) z^{-j} = \frac{1}{z} \sum_{j=1}^{\infty} x(j-1) z^{-(j-1)} = \frac{1}{z} \hat{x}(z).$$

In particular we note that the transfer function of $\mathbf{S}$ is given by $1/z$.

(b) Another routine calculation, using the fact that $\Delta_0 = I - \mathbf{S}$, shows the transfer function of Δ_0 is given by $(z-1)/z$.

(c) Using the fact that by Lemma 2.1.2 part (a) $\Delta_0 J = \mathbf{S}$, and (a) and (b), we conclude that the transfer function of J is given by $1/(z-1)$.

(d) Using the fact that by Lemma 2.1.2 part (a) $\Delta_0 J_0 = I$, and (b), we conclude that the transfer function of J_0 is given by $z/(z-1)$. $\diamond$

3.4 Discrete-time steady-state gains and step error

Let $\mathbf{G} \in H^\infty(\mathbb{E}_1, \mathscr{B}(U))$ denote the transfer function of a shift-invariant operator $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$. By shift-invariance, G is causal, and therefore G extends to a shift-invariant operator from $F(\mathbb{Z}_+, U)$ into itself as follows: Let $u \in F(\mathbb{Z}_+, U)$. We extend G to an operator $F(\mathbb{Z}_+, U) \to F(\mathbb{Z}_+, U)$ by setting

$$(Gu)(n) = (G(\mathbf{P}_N u))(n), \qquad n \in \underline{N},$$

where $\underline{N} := \{0, 1, \ldots, N\}$ and $N \in \mathbb{Z}_+$. By causality of G this definition yields a well-defined extension of G to $F(\mathbb{Z}_+, U)$. We shall use the same symbol G to denote the original operator on $l^2(\mathbb{Z}_+, U)$ and its shift-invariant extension to $F(\mathbb{Z}_+, U)$.

Definition. If there exists an operator $\Gamma \in \mathscr{B}(U)$ such that

$$\lim_{n \to \infty} (G(\vartheta \xi))(n) = \Gamma \xi, \quad \forall \xi \in U,$$

then we say that Γ is the *asymptotic steady-state gain* of G. Moreover, if there exists an operator $\Gamma \in \mathscr{B}(U)$ such that

$$G(\vartheta\xi) - \vartheta\Gamma\xi \in l^2(\mathbb{Z}_+, U), \quad \forall \xi \in U,$$

then Γ is said to be the *l^2-steady-state gain* of G. If the asymptotic steady-state gain or the l^2-steady-state gain of G exist, then, for $\xi \in U$, the function

$$\sigma^\xi := G(\vartheta\xi) - \vartheta\Gamma\xi$$

is said to be the *step error* associated with ξ.

The asymptotic steady-state gain and the l^2-steady-state gain may or may not exist. The existence of the l^2-steady-state gain implies the existence of the asymptotic steady-state gain. The converse is not true. If they both exist, then they coincide.

Trivially, under the additional assumption that G is the input-output operator of a finite-dimensional l^2-stable state-space system (i.e., $\mathbf{G}$ is rational), the asymptotic steady-state gain and the l^2-steady-state gain exist and are given by $\mathbf{G}(1)$; furthermore, there exist $M > 0$ and $\beta \in (0, 1)$ such that $\|\sigma^\xi(n)\| \leq M\beta^n\|\xi\|$ for all $n \in \mathbb{Z}_+$ and for all $\xi \in U = \mathbb{R}^m$.

Throughout the thesis we shall often impose the following assumption on $\mathbf{G}$.

(A) There exists $\Gamma \in \mathscr{B}(U)$ such that

$$\limsup_{z \to 1, z \in \mathbb{E}_1} \left\| \frac{1}{z-1}(\mathbf{G}(z) - \Gamma) \right\| < \infty. \tag{3.7}$$

Remark 3.4.1. If $\mathbf{G}$ extends analytically into a neighbourhood of 1 (which in particular is the case if $G \in \mathscr{B}(l^2_\beta(\mathbb{Z}_+, U))$ for some $\beta \in (0, 1)$), then (A) holds with $\Gamma = \mathbf{G}(1)$. Furthermore, if $\mathbf{G}$ is the transfer function of a strongly stable discrete-time state-space system (see Chapter 6) with the additional property that 1 is in the resolvent set of the generator of the discrete-time semigroup (which is trivially true for power-stable systems), then (A) holds. $\diamond$

Note that, by assumption (A),

$$z \mapsto \widehat{\sigma}^\xi(z) = \frac{z}{z-1}(\mathbf{G}(z) - \Gamma)\xi \in H^\infty(\mathbb{E}_1, \mathscr{B}(U)), \qquad \forall\, \xi \in U.$$

Consequently, by Theorem 3.3.2 part (a),

$$\|\sigma^\xi\|_{l^2} = \|\widehat{\sigma}^\xi(z)\|_{H^2} \leq \|\widehat{\sigma}^\xi(z)\|_{H^\infty} \leq \gamma\|\xi\|, \qquad \forall \xi \in U,$$

where $\gamma := \sup_{z\in\mathbb{E}_1} \|(z/(z-1))(\mathbf{G}(z)-\Gamma)\|$, showing that the operator $\Sigma : \xi \mapsto \sigma^\xi$ is in $\mathscr{B}(U, l^2(\mathbb{Z}_+, U))$. In particular, if assumption (A) holds, then Γ is the l^2-steady-state gain of G. Using Remark 3.4.1, it can be easily shown that, if $G \in \mathscr{B}(l^2_\beta(\mathbb{Z}_+, U))$ for some $\beta \in (0,1)$, then $\Sigma \in \mathscr{B}(U, l^2_\beta(\mathbb{Z}_+, U))$, in which case we have

$$\|\sigma^\xi(n)\|\beta^{-n} \leq \|\sigma^\xi\|_{l^2_\beta(\mathbb{Z}_+,U)} = \|\Sigma\xi\|_{l^2_\beta(\mathbb{Z}_+,U)} \leq \|\Sigma\|\|\xi\|, \quad \forall\, n \in \mathbb{Z}_+,\ \forall\, \xi \in U,$$

(where $\|\Sigma\|$ denotes the operator norm of Σ), showing that

$$\|\sigma^\xi(n)\| \leq \|\Sigma\|\beta^n\|\xi\|, \quad \forall\, n \in \mathbb{Z}_+,\ \forall\, \xi \in U.$$

The following result gives a time-domain characterization of assumption (A).

Lemma 3.4.2. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ be shift-invariant with transfer function $\mathbf{G}$ and let $\Gamma \in \mathscr{B}(U)$. Then* (3.7) *holds* (*i.e., $\mathbf{G}$ satisfies assumption* (A)) *if and only if $GJ - \Gamma J \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$.*

Proof. ($\Rightarrow$) Suppose that $\mathbf{G}$ satisfies assumption (A). Consider the operator $GJ-\Gamma J$. By shift-invariance of G and J it follows that $GJ-\Gamma J$ is shift-invariant and its transfer function is given by

$$\frac{1}{z-1}(\mathbf{G}(z) - \Gamma), \qquad z \in \mathbb{E}_1.$$

From assumption (A) and the fact that $\mathbf{G} \in H^\infty(\mathbb{E}_1, \mathscr{B}(U))$, we conclude that, $z \mapsto (\mathbf{G}(z) - \Gamma)/(z-1) \in H^\infty(\mathbb{E}_1, \mathscr{B}(U)) \subseteq H^2(\mathbb{E}_1, \mathscr{B}(U))$ and so, by Theorem 3.3.2 (a), $GJ - \Gamma J \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$.

($\Leftarrow$) Suppose $GJ - \Gamma J \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$. As before, since G and J are shift-invariant, it follows that $GJ - \Gamma J$ is shift-invariant. Consequently, $GJ - \Gamma J$ has a $H^\infty(\mathbb{E}_1, \mathscr{B}(U))$ transfer function. Since this transfer function is given by

$$\frac{1}{z-1}(\mathbf{G}(z) - \Gamma), \qquad z \in \mathbb{E}_1,$$

it follows that assumption (A) holds. $\square$

Remark 3.4.3. Note that Lemma 3.4.2 remains true if J is replaced by J_0. $\diamond$

Furthermore, we have the following result on the behaviour of Gu for converging inputs u.

Proposition 3.4.4. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ be shift-invariant with transfer function $\mathbf{G}$. Assume that* (A) *is satisfied.*

(a) *If* $u \in F(\mathbb{Z}_+, U)$ *is such that* $u - \vartheta u^\infty \in l^2(\mathbb{Z}_+, U)$ *for some* $u^\infty \in U$*, then* $Gu - \vartheta \Gamma u^\infty \in l^2(\mathbb{Z}_+, U)$.

(b) *If* $u \in F(\mathbb{Z}_+, U)$ *is such that* $\Delta_0 u \in l^2(\mathbb{Z}_+, U)$ *and* $u^\infty := \lim_{n\to\infty} u(n)$ *exists, then* $\lim_{n\to\infty}(Gu)(n) = \Gamma u^\infty$.

Remarks 3.4.5. (i) Note that if $u - \vartheta u^\infty \in l^2(\mathbb{Z}_+, U)$ for some $u^\infty \in U$, then $\lim_{n\to\infty} u(n) = u^\infty$ and $\Delta_0 u \in l^2(\mathbb{Z}_+, U)$. Furthermore, it is easy to see that there exist sequences $u \in F(\mathbb{Z}_+, U)$ such that $u^\infty := \lim_{n\to\infty} u(n)$ exists and $\Delta_0 u \in l^2(\mathbb{Z}_+, U)$, but $u - \vartheta u^\infty \notin l^2(\mathbb{Z}_+, U)$, for example, with $U = \mathbb{R}$, $u(n) = u^\infty + 1/\sqrt{n+1}$, for all $n \in \mathbb{Z}_+$ and some $u^\infty \in \mathbb{R}$. This shows that the hypothesis on u in part (b) is 'strictly weaker' than that in part (a).

(ii) Noting that $\Delta_0 u = \mathbf{S}\Delta u + \delta u(0)$, Proposition 3.4.4 remains true with Δ_0 replaced by Δ. ◇

Proof of Proposition 3.4.4. (a) Note that,

$$Gu - \vartheta\Gamma u^\infty = G(u - \vartheta u^\infty) + G\vartheta u^\infty - \vartheta \Gamma u^\infty. \tag{3.8}$$

Since by assumption $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ and $u - \vartheta u^\infty \in l^2(\mathbb{Z}_+, U)$ it follows that $G(u - \vartheta u^\infty) \in l^2(\mathbb{Z}_+, U)$. Furthermore, since assumption (A) holds, Γ is the l^2-steady-state gain of G and so $G\vartheta u^\infty - \vartheta\Gamma u^\infty \in l^2(\mathbb{Z}_+, U)$. Consequently, it follows from (3.8) that $Gu - \vartheta\Gamma u^\infty \in l^2(\mathbb{Z}_+, U)$.

(b) Setting $H := GJ_0 - \Gamma J_0$, it follows from the shift-invariance of G that,

$$H\Delta_0 u = Gu - \Gamma u. \tag{3.9}$$

By assumption (A) and Lemma 3.4.2 we have that $H \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$. Consequently, since by assumption $\Delta_0 u \in l^2(\mathbb{Z}_+, U)$, we have

$$\lim_{n\to\infty} (H(\Delta_0 u))(n) = 0.$$

Hence it follows from (3.9) that

$$\lim_{n\to\infty} (Gu)(n) = \lim_{n\to\infty} \Gamma u(n) = \Gamma u^\infty.$$

□

Occasionally, it will also be necessary to impose the following assumption on $\mathbf{G}$.

($\mathbf{A}'$) There exists $\Gamma, \Gamma' \in \mathscr{B}(U)$ such that

$$\limsup_{z\to 1, z\in \mathbb{E}_1} \left\| \frac{1}{(z-1)^2}(\mathbf{G}(z) - \Gamma - (z-1)\Gamma') \right\| < \infty. \tag{3.10}$$

Remarks 3.4.6. (i) Note that if $\mathbf{G}$ satisfies assumption (A′) (i.e., (3.10) holds), then $\mathbf{G}$ satisfies assumption (A) (i.e., (3.7) holds).

(ii) If $\mathbf{G}$ extends analytically into a neighbourhood of 1 (which in particular is the case if $G \in \mathscr{B}(l^2_\beta(\mathbb{Z}_+, U))$ for some $\beta \in (0,1)$), then (A′) holds with $\Gamma = \mathbf{G}(1)$ and $\Gamma' = \mathbf{G}'(1) = \lim_{z \to 1}(\mathbf{G}(z) - \mathbf{G}(1))/(z-1)$. Consequently, if $\mathbf{G}$ is the transfer function of a strongly stable discrete-time state-space system (see Chapter 6) with the additional property that 1 is in the resolvent set of the generator of the discrete-time semigroup (which is trivially true for power-stable systems), then (A′) holds. ◇

The following result gives a time-domain characterization of assumption (A′).

Lemma 3.4.7. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ be shift-invariant with transfer function $\mathbf{G}$ and let $\Gamma, \Gamma' \in \mathscr{B}(U)$. Then* (3.10) *holds (i.e., $\mathbf{G}$ satisfies assumption* (A′)*) if and only if $(GJ - \Gamma J)J - \Gamma' J \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$.*

Proof. ($\Rightarrow$) Suppose that $\mathbf{G}$ satisfies assumption (A′). Consider the operator $(GJ - \Gamma J)J - \Gamma' J$. By shift-invariance of G and J it follows that $(GJ - \Gamma J)J - \Gamma' J$ is shift-invariant and its transfer function is given by

$$\frac{1}{(z-1)^2}\Big(\mathbf{G}(z) - \Gamma - (z-1)\Gamma'\Big), \qquad z \in \mathbb{E}_1. \tag{3.11}$$

From assumption (A′) and the fact that $\mathbf{G} \in H^\infty(\mathbb{E}_1, \mathscr{B}(U))$, we conclude that,

$$z \mapsto \frac{1}{(z-1)^2}\Big(\mathbf{G}(z) - \Gamma - (z-1)\Gamma'\Big) \in H^\infty(\mathbb{E}_1, \mathscr{B}(U)) \subseteq H^2(\mathbb{E}_1, \mathscr{B}(U))$$

and so, by Theorem 3.3.2 (a), $(GJ - \Gamma J)J - \Gamma' J \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$.

($\Leftarrow$) Suppose $(GJ - \Gamma J)J - \Gamma' J \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$. As before, since G and J are shift-invariant, it follows that $(GJ - \Gamma J)J - \Gamma' J$ is shift-invariant. Consequently, $(GJ - \Gamma J)J - \Gamma' J$ has a $H^\infty(\mathbb{E}_1, \mathscr{B}(U))$ transfer function. Since this transfer function is given by (3.11), it follows that assumption (A′) holds. □

Remark 3.4.8. Note that Lemma 3.4.7 remains true if J is replaced by J_0. ◇

3.5 Existence and uniqueness results

In this section we discuss existence and uniqueness results for general discrete-time equations with non-linearities. Consider the following equation

$$u = r - J(G(\varphi \circ u)), \tag{3.12}$$

where $r : \mathbb{Z}_+ \to U$ is a given forcing function, $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ is shift-invariant, $\varphi : \mathbb{Z}_+ \times U \to U$ is a time-dependent non-linearity and $\varphi \circ u$ denotes the function $n \mapsto \varphi(n, u(n))$. Recall that by shift-invariance, G is causal and therefore G extends to a shift-invariant operator from $F(\mathbb{Z}_+, U)$ into itself. A solution of (3.12) is a function $u \in F(\mathbb{Z}_+, U)$ satisfying (3.12). Recall the following notation: for $N \in \mathbb{Z}_+$, let $\underline{N} := \{0, 1, \ldots, N\}$. We have the following result on existence and uniqueness of solutions of (3.12).

Proposition 3.5.1. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ be shift-invariant, $\varphi : \mathbb{Z}_+ \times U \to U$ be a time-dependent non-linearity and $r : \mathbb{Z}_+ \to U$ be a given forcing function. Then* (3.12) *has a unique solution $u : \mathbb{Z}_+ \to U$.*

Proof. For $N \in \mathbb{Z}_+$ we define $G_N : F(\underline{N}, U) \to F(\underline{N}, U)$ by

$$(G_N v)(n) := (G v_N)(n), \qquad n \in \underline{N},\ v \in F(\underline{N}, U) \tag{3.13}$$

where

$$v_N(n) = \begin{cases} v(n), & n \in \underline{N}, \\ 0, & n > N. \end{cases} \tag{3.14}$$

Hence, by causality of G, for each $N \in \mathbb{Z}_+$ and $v \in F(\mathbb{Z}_+, U)$,

$$(Gv)(n) = (G_N v|_{\underline{N}})(n), \qquad n \in \underline{N}.$$

Define recursively,

$$\begin{aligned} u(0) &:= r(0) \\ u(1) &:= r(1) - (G_0(\varphi \circ u))(0) \\ u(2) &:= r(2) - \sum_{j=0}^{1} (G_1(\varphi \circ u))(j) \\ &\ \vdots \\ u(n+1) &:= r(n+1) - \sum_{j=0}^{n} (G_n(\varphi \circ u))(j). \end{aligned}$$

Then u solves (3.12). Furthermore, it is clear by the recursive construction that u is unique: if v is a solution of (3.12), then $v(0) = r(0) = u(0)$, hence,

$$v(1) = r(1) - (G_0(\varphi \circ v))(0) = r(1) - (G_0(\varphi \circ u))(0) = u(1)$$

and so on iteratively. $\square$

We now introduce a discrete-time integrator with direct feedthrough. Consider the following discrete-time equation,

$$u = r - J_0(G(\varphi \circ u)), \tag{3.15}$$

where φ, G and r are as before. A solution of (3.15) is a function $u \in F(\mathbb{Z}_+, X)$ satisfying (3.15)

Example 3.5.2. The following example shows that an equation of the form (3.15) does not always have a solution. Let $U = \mathbb{R}$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by,

$$\varphi(\xi) := \begin{cases} \xi + 1, & \text{if } \xi < -1, \\ 0, & \text{if } \xi \in [-1, 1], \\ \xi - 1, & \text{if } \xi > 1, \end{cases}$$

set $G = -I$ and let $r = 2\vartheta$. Then, by (3.15),

$$u(n) = 2 + \sum_{j=0}^{n} \varphi(u(j)), \qquad \forall\, n \in \mathbb{Z}_+. \tag{3.16}$$

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(\xi) := \xi - \varphi(\xi)$ for all $\xi \in \mathbb{R}$, that is,

$$f(\xi) := \begin{cases} -1, & \text{if } \xi < -1, \\ \xi, & \text{if } \xi \in [-1, 1], \\ 1, & \text{if } \xi > 1. \end{cases}$$

It follows from (3.16) that

$$f(u(0)) = u(0) - \varphi(u(0)) = 2.$$

Noting that $2 \notin \operatorname{im} f$ we see that (3.16) does not have a solution. $\diamond$

The following result gives a condition under which (3.15) has solutions and moreover, provided it exists, when a solution is unique.

Proposition 3.5.3. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ be shift-invariant, $\varphi : \mathbb{Z}_+ \times U \to U$ be a time-dependent non-linearity and $r : \mathbb{Z}_+ \to U$ be a given forcing function. Define $\mathbf{G}(\infty) := \lim_{|z|\to\infty} \mathbf{G}(z)$. Then, there exists at least one solution (a unique solution, respectively) of (3.15) if, for every $n \in \mathbb{Z}_+$, the map $f_n : U \to U$ defined by*

$$f_n(\xi) := \xi + \mathbf{G}(\infty)\varphi(n, \xi), \quad \forall\, (n, \xi) \in \mathbb{Z}_+ \times U, \tag{3.17}$$

is surjective (bijective, respectively).

Remark 3.5.4. Note that if $\mathbf{G}(\infty) = 0$ then $f_n = I$ for all $n \in \mathbb{Z}_+$ and (3.15) has a unique solution. $\diamond$

Proof of Proposition 3.5.3. Define $M : F(\mathbb{Z}_+, U) \to F(\mathbb{Z}_+, U)$ by $M := J_0 G$. It follows from (3.15) that

$$u + M(\varphi \circ u) = r. \tag{3.18}$$

By linearity and shift-invariance of G and J_0, M is also linear and shift-invariant. Consequently, by Proposition 3.1.2, M is a convolution operator with convolution kernel $k \in F(\mathbb{Z}_+, \mathscr{B}(U))$, that is,

$$(Mu)(n) = (k * u)(n) = \sum_{j=0}^{n} k(n-j)u(j).$$

Defining $\widetilde{M} := M - k(0)I$, (which is also a convolution operator) from (3.18) we have,

$$u(n) + k(0)(\varphi \circ u)(n) = r(n) - (\widetilde{M}(\varphi \circ u))(n). \tag{3.19}$$

Since r is given, the RHS of (3.19) can be determined by only knowing the values $u(0), \dots, u(n-1)$ but, $u(n)$ could still explicitly occur in the term $k(0)(\varphi \circ u)(n)$. Denoting the transfer function of M by $\mathbf{M}$ we have,

$$\mathbf{M}(z) = \frac{z}{z-1}\mathbf{G}(z), \qquad z \in \mathbb{E}_1.$$

Clearly,

$$k(0) = \lim_{|z| \to \infty} \mathbf{M}(z) = \lim_{|z| \to \infty} \mathbf{G}(z) =: \mathbf{G}(\infty).$$

It follows from (3.19) that

$$f_n(u(n)) = r(n) - (\widetilde{M}(\varphi \circ u))(n),$$

where f_n is defined by (3.17). Suppose now that for all $n \in \mathbb{Z}_+$, the map f_n is surjective. Denoting the preimage of f_n by f_n^{-1} for all $n \in \mathbb{Z}_+$, we can construct a solution of (3.15) as follows: We define $u \in F(\mathbb{Z}_+, U)$ recursively by,

$$\begin{aligned}
u(0) &\in f_0^{-1}(r(0) - (\widetilde{M}_0(\varphi \circ u))(0)) \\
u(1) &\in f_1^{-1}(r(1) - (\widetilde{M}_1(\varphi \circ u))(1)) \\
&\vdots \\
u(n) &\in f_n^{-1}(r(n) - (\widetilde{M}_n(\varphi \circ u))(n))
\end{aligned}$$

where, for $N \in \mathbb{Z}_+$, $\widetilde{M}_N : F(\underline{N}, U) \to F(\underline{N}, U)$ is defined by

$$(\widetilde{M}_N v)(n) := (\widetilde{M} v_N)(n), \quad n \in \underline{N}, \; v \in F(\underline{N}, U),$$

and v_N is given by (3.14). If, further, f_n is injective for each $n \in \mathbb{Z}_+$ then the preimage of $f_n^{-1}(\xi)$, $\xi \in U$, consists of exactly one point and we see that the solution constructed above is unique. $\square$

3.6 Notes and references

Most of the results in this chapter are well known. Whilst difficult to locate, the proof of Lemma 3.1.1 is standard and included for completeness. Proposition 3.1.2 is well known, however, in many cases in the literature it is stated for the special case $X = \mathbb{R}$. A proof of Proposition 3.1.2 in the general Banach space setting is difficult to locate in the available literature, consequently, we include a proof for completeness. We remark that the existence of a transfer function for a bounded linear shift-invariant operator on $l^2(\mathbb{Z}_+, U)$ (see Theorem 3.3.4), is essential for the development of the thesis. We note that Lemma 3.4.2, which gives a time-domain characterization of assumption (A), is a key observation which will be required throughout the thesis. Proposition 3.4.4 is new and in the context of this thesis is important in studying low-gain integral control of discrete-time systems subject to input and output non-linearities, (see Chapter 5). The existence and uniqueness results in §3.5 seem to be new, the proofs of which are fairly routine.

Chapter 4

Absolute stability results for infinite-dimensional discrete-time systems

In this chapter we present input-output absolute stability results for discrete-time systems. We derive stability results of Popov-type and circle-criterion type.

We begin by considering an absolute stability problem for the feedback system shown in Figure 4.1. The input-output operator G is linear, shift-invariant and bounded from $l^2(\mathbb{Z}_+, U)$ into itself, $\varphi : \mathbb{Z}_+ \times U \to U$ is a time dependent nonlinearity and $r : \mathbb{Z}_+ \to U$ is the input of the feedback system (or forcing function).

From Figure 4.1 we can derive the following governing equations,

$$u = r - y, \qquad v = G(\varphi \circ u), \qquad y = Jv,$$

where $\varphi \circ u$ denotes the function $n \mapsto \varphi(n, u(n))$. Equivalently,

$$u = r - (JG)(\varphi \circ u). \tag{4.1}$$

It follows from Lemma 3.5.1 that there exists a unique solution u to (4.1). Throughout this chapter we impose assumption (A) on $\mathbf{G}$, the transfer function of G, with $\Gamma = \mathbf{G}(1) := \lim_{z \to 1, z \in \mathbf{E}_1} \mathbf{G}(z)$. Note that the existence of $\lim_{z \to 1, z \in \mathbf{E}_1} \mathbf{G}(z)$ is implied by imposing assumption (A).

4.1 Absolute stability results of Popov-type

We require the following lemma the proof of which can be found in Appendix 1.

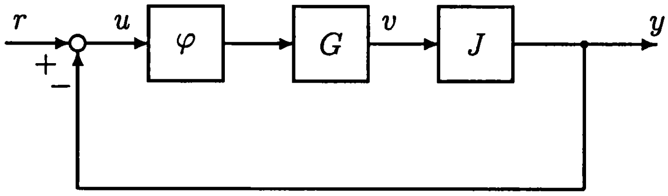

Figure 4.1: Discrete-time feedback system with J integrator

Lemma 4.1.1. *For $v \in F(\mathbb{Z}_+, U)$, we have the following formula,*

$$\operatorname{Re} \sum_{n=1}^{m} \left\langle v(n), \sum_{k=0}^{n-1} v(k) \right\rangle = \frac{1}{2} \left\| \sum_{k=0}^{m} v(k) \right\|^2 - \frac{1}{2} \sum_{k=0}^{m} \|v(k)\|^2, \qquad \forall\, m \in \mathbb{N}.$$

Throughout the rest of this section we assume that $U = \mathbb{R}$, that φ is time-independent and consider the absolute stability problem shown in Figure 4.1. The following result is a stability criterion of Popov-type in an input-output context.

Theorem 4.1.2. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ be a shift-invariant operator with transfer function $\mathbf{G}$ satisfying assumption (A) and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a measurable non-decreasing non-linearity. Assume that $\mathbf{G}(1) > 0$ and there exists numbers $q \geq 0$, $\varepsilon > 0$ and $a \in (0, \infty)$ such that*

$$\varphi(v)v \geq \frac{1}{a}\varphi^2(v), \qquad \forall\, v \in \mathbb{R}, \tag{4.2}$$

and

$$\frac{1}{a} + \operatorname{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta} - 1}\right)\mathbf{G}(e^{i\theta})\right] \geq \varepsilon, \qquad \text{a.a. } \theta \in (0, 2\pi). \tag{4.3}$$

Let $r \in m^2(\mathbb{Z}_+, \mathbb{R})$ and let $u : \mathbb{Z}_+ \to \mathbb{R}$ be the unique solution of (4.1). Then the following statements hold.

1. *There exists a constant K (which depends only on q, ε, a and G, but not on r) such that,*

$$\begin{aligned} \|u\|_{l^\infty} + \|\Delta_0 u\|_{l^2} + \|\varphi \circ u\|_{l^2} + (\|(\varphi \circ u)u\|_{l^1})^{1/2} \\ + \sup_{n \geq 0} \left| \sum_{j=0}^{n} (\varphi \circ u)(j) \right| \leq K \|r\|_{m^2}. \end{aligned} \tag{4.4}$$

2. The limits

$$u^\infty := \lim_{n\to\infty} u(n), \qquad \lim_{n\to\infty}\sum_{j=0}^{n}(\varphi\circ u)(j), \tag{4.5}$$

exist, are finite and, if φ is continuous, $\varphi(u^\infty)=0$.

Proof. We have,

$$u + (JG)(\varphi\circ u) = r, \tag{4.6}$$

or equivalently,

$$\Delta_0 u + \mathbf{S}G(\varphi\circ u) = \Delta_0 r. \tag{4.7}$$

We now write (4.6) in a slightly more convenient form, namely,

$$u + H(\varphi\circ u) + \mathbf{G}(1)J(\varphi\circ u) = r, \tag{4.8}$$

where $H := J\circ G - \mathbf{G}(1)J$. It is clear that this operator is shift-invariant with transfer function $\mathbf{H}$ given by,

$$\mathbf{H}(z) = \frac{1}{z-1}[\mathbf{G}(z) - \mathbf{G}(1)], \qquad z\in\mathbb{E}_1. \tag{4.9}$$

From assumption (A) and from the fact that $\mathbf{G}\in H^\infty(\mathbb{E}_1)$, we conclude that $\mathbf{H}\in H^\infty(\mathbb{E}_1)$, and hence $H\in\mathscr{B}(l^2(\mathbb{Z}_+,\mathbb{R}))$.

We multiply (4.7) by q and to this add (4.8) to obtain

$$\begin{aligned} q(\Delta_0 u)(j) + u(j) \,+\, &G_q(\varphi\circ u)(j) + \mathbf{G}(1)(J(\varphi\circ u))(j)\\ &= q(\Delta_0 r)(j) + r(j), \qquad \forall\, j\in\mathbb{Z}_+, \end{aligned} \tag{4.10}$$

where we have defined the operator G_q by

$$G_q := q\mathbf{S}G + H.$$

Invoking (4.9), we see that the transfer function $\mathbf{G}_q$ of G_q is given by

$$\mathbf{G}_q(z) := \frac{q}{z}\mathbf{G}(z) + \frac{1}{z-1}[\mathbf{G}(z) - \mathbf{G}(1)], \qquad z\in\mathbb{E}_1.$$

Multiplying through by $(\varphi \circ u)(j)$ and summing from 0 to n in (4.10) yields,

$$
\begin{aligned}
q\sum_{j=0}^{n}(\varphi \circ u)(j)(\Delta_0 u)(j) + \sum_{j=0}^{n}(\varphi \circ u)(j)u(j)& \\
+\sum_{j=0}^{n}(\varphi \circ u)(j)(G_q(\varphi \circ u))(j) + \mathbf{G}(1)\sum_{j=1}^{n}(\varphi \circ u)(j)\sum_{k=0}^{j-1}(\varphi \circ u)(k)& \\
= q\sum_{j=0}^{n}(\varphi \circ u)(j)(\Delta_0 r)(j) + \sum_{j=0}^{n}(\varphi \circ u)(j)r(j).& \qquad (4.11)
\end{aligned}
$$

An application of Lemma 4.1.1 to the last term on the LHS of (4.11) yields,

$$
\begin{aligned}
\mathbf{G}(1)\sum_{j=1}^{n}(\varphi \circ u)(j)\sum_{k=0}^{j-1}(\varphi \circ u)(k)& \\
= \frac{\mathbf{G}(1)}{2}\left[\left(\sum_{j=0}^{n}(\varphi \circ u)(j)\right)^2 - \sum_{j=0}^{n}(\varphi \circ u)^2(j)\right].& \qquad (4.12)
\end{aligned}
$$

Combining (4.12) with (4.11) gives,

$$
\begin{aligned}
q\sum_{j=0}^{n}(\varphi \circ u)(j)(\Delta_0 u)(j) + \sum_{j=0}^{n}(\varphi \circ u)(j)u(j) + \sum_{j=0}^{n}(\varphi \circ u)(j)(G_q(\varphi \circ u))(j)& \\
+\frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi \circ u)(j)\right)^2 - \frac{\mathbf{G}(1)}{2}\sum_{j=0}^{n}(\varphi \circ u)^2(j)& \\
= q\sum_{j=0}^{n}(\varphi \circ u)(j)(\Delta_0 r)(j) + \sum_{j=0}^{n}(\varphi \circ u)(j)r(j).& \qquad (4.13)
\end{aligned}
$$

We note that,

$$
\begin{aligned}
\operatorname{Re}\mathbf{G}_q(z) &= \operatorname{Re}\left(\frac{q}{z}\mathbf{G}(z) + \frac{1}{z-1}[\mathbf{G}(z) - \mathbf{G}(1)]\right) \\
&= \operatorname{Re}\left[\left(\frac{q}{z} + \frac{1}{z-1}\right)\mathbf{G}(z)\right] - \mathbf{G}(1)\operatorname{Re}\frac{1}{z-1}.
\end{aligned}
$$

Taking $z = e^{i\theta}$ with $\theta \in (0, 2\pi)$,

$$\operatorname{Re}\frac{1}{z-1} = \operatorname{Re}\left(\frac{\overline{z-1}}{|z-1|^2}\right) = \operatorname{Re}\left(\frac{\overline{e^{i\theta}-1}}{|e^{i\theta}-1|^2}\right) = \frac{\cos\theta - 1}{(\cos\theta-1)^2 + \sin^2\theta}$$
$$= \frac{\cos\theta - 1}{2(1-\cos\theta)}$$
$$= -\frac{1}{2},$$

and we obtain,

$$\operatorname{Re}\mathbf{G}_q(e^{i\theta}) = \operatorname{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right] + \frac{\mathbf{G}(1)}{2}, \qquad \text{a.a. } \theta \in (0, 2\pi). \quad (4.14)$$

Combining (4.14) with (4.3) we see that,

$$\begin{aligned}\operatorname{Re}\mathbf{G}_q(e^{i\theta}) &= \operatorname{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right] + \frac{\mathbf{G}(1)}{2} \\ &\geq \varepsilon - \frac{1}{a} + \frac{\mathbf{G}(1)}{2}. \end{aligned} \tag{4.15}$$

Define $v : \mathbb{Z} \to \mathbb{R}$ by

$$v(j) = \begin{cases} (\varphi \circ u)(j), & \text{if } 0 \leq j \leq n, \\ 0, & \text{otherwise.} \end{cases} \tag{4.16}$$

Applying Theorem 3.3.3 and taking real parts,

$$\begin{aligned}\sum_{j=0}^{\infty} v(j)(G_q v)(j) &= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left[\hat{v}(e^{i\theta})\overline{(\widehat{G_q v})(e^{i\theta})}\right] d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} |\hat{v}(e^{i\theta})|^2 \operatorname{Re}\overline{\mathbf{G}_q(e^{i\theta})}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} |\hat{v}(e^{i\theta})|^2 \operatorname{Re}\mathbf{G}_q(e^{i\theta})\, d\theta. \end{aligned} \tag{4.17}$$

Using (4.15) and applying again Theorem 3.3.3, we obtain,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |\hat{v}(e^{i\theta})|^2 \operatorname{Re}\mathbf{G}_q(e^{i\theta})\, d\theta &\geq \frac{1}{2\pi}\left(\varepsilon - \frac{1}{a} + \frac{\mathbf{G}(1)}{2}\right)\int_0^{2\pi} |\hat{v}(e^{i\theta})|^2\, d\theta \\ &= \left(\varepsilon - \frac{1}{a} + \frac{\mathbf{G}(1)}{2}\right)\sum_{j=0}^{\infty} v^2(j). \end{aligned}$$

Invoking (4.16) and (4.17), we conclude that,

$$\sum_{j=0}^{n}(\varphi\circ u)(j)(G_q(\varphi\circ u))(j) \geq \left(\varepsilon - \frac{1}{a} + \frac{\mathbf{G}(1)}{2}\right)\sum_{j=0}^{n}(\varphi\circ u)^2(j). \tag{4.18}$$

Applying (4.18) to (4.13) gives,

$$\begin{aligned}
&q\sum_{j=0}^{n}(\varphi\circ u)(j)(\triangle_0 u)(j) + \sum_{j=0}^{n}(\varphi\circ u)(j)u(j)\\
&\quad+\left(\varepsilon - \frac{1}{a}\right)\sum_{j=0}^{n}(\varphi\circ u)^2(j) + \frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j)\right)^2\\
&\qquad\leq q\sum_{j=0}^{n}(\varphi\circ u)(j)(\triangle_0 r)(j) + \sum_{j=0}^{n}(\varphi\circ u)(j)r(j).
\end{aligned} \tag{4.19}$$

Define $\Phi(v) := \int_0^v \varphi(\sigma)\,d\sigma$ for all $v \in \mathbb{R}$. Using the fact that φ is non-decreasing we can estimate the first term on the LHS of (4.19) as follows,

$$\begin{aligned}
\sum_{j=0}^{n}(\varphi\circ u)(j)(\triangle_0 u)(j) &\geq \sum_{j=1}^{n}\int_{u(j-1)}^{u(j)}\varphi(\sigma)\,d\sigma + \int_0^{u(0)}\varphi(\sigma)\,d\sigma\\
&= \int_0^{u(n)}\varphi(\sigma)\,d\sigma\\
&= \Phi(u(n)), \qquad \forall\, n \in \mathbb{Z}_+.
\end{aligned}$$

Using this estimate in (4.19) we obtain,

$$\begin{aligned}
&q\Phi(u(n)) + \sum_{j=0}^{n}(\varphi\circ u)(j)u(j)\\
&\quad+\left(\varepsilon - \frac{1}{a}\right)\sum_{j=0}^{n}(\varphi\circ u)^2(j) + \frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j)\right)^2\\
&\qquad\leq q\sum_{j=0}^{n}(\varphi\circ u)(j)(\triangle_0 r)(j) + \sum_{j=0}^{n}(\varphi\circ u)(j)r(j).
\end{aligned} \tag{4.20}$$

By assumption $r = r_1 + r_2\vartheta$, where $r_1 \in l^2(\mathbb{Z}_+, \mathbb{R})$ and $r_2 \in \mathbb{R}$. Using this decomposition of r we can write the last term on the RHS of (4.20) as,

$$\sum_{j=0}^{n}(\varphi\circ u)(j)r(j) = \sum_{j=0}^{n}(\varphi\circ u)(j)r_1(j) + \sum_{j=0}^{n}(\varphi\circ u)(j)r_2. \tag{4.21}$$

Using (4.21) in (4.20) gives,

$$
\begin{aligned}
&q\Phi(u(n)) + \sum_{j=0}^{n}(\varphi\circ u)(j)u(j) + \left(\varepsilon - \frac{1}{a}\right)\sum_{j=0}^{n}(\varphi\circ u)^2(j) \\
&\quad + \frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j)\right)^2 - \sum_{j=0}^{n}(\varphi\circ u)(j)r_2 \\
&\quad \le q\sum_{j=0}^{n}(\varphi\circ u)(j)(\triangle_0 r)(j) + \sum_{j=0}^{n}(\varphi\circ u)(j)r_1(j). \qquad (4.22)
\end{aligned}
$$

Completing the square gives,

$$
\begin{aligned}
&\frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j)\right)^2 - \sum_{j=0}^{n}(\varphi\circ u)(j)r_2 \\
&\quad = \frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)^2 - \frac{1}{2}[\mathbf{G}(1)]^{-1}r_2^2. \qquad (4.23)
\end{aligned}
$$

So using (4.23) in (4.22) yields the following,

$$
\begin{aligned}
&q\Phi(u(n)) + \sum_{j=0}^{n}(\varphi\circ u)(j)u(j) + \left(\varepsilon - \frac{1}{a}\right)\sum_{j=0}^{n}(\varphi\circ u)^2(j) \\
&\quad + \frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)^2 - \frac{1}{2}[\mathbf{G}(1)]^{-1}{r_2}^2 \\
&\quad \le q\sum_{j=0}^{n}(\varphi\circ u)(j)(\triangle_0 r)(j) + \sum_{j=0}^{n}(\varphi\circ u)(j)r_1(j). \qquad (4.24)
\end{aligned}
$$

Now the RHS of (4.24) can be estimated as follows (since $ab \le \frac{1}{2}\varepsilon a^2 + b^2/(2\varepsilon)$ for non-negative numbers a and b)

$$
\begin{aligned}
&\sum_{j=0}^{n}|(\varphi\circ u)(j)[q(\triangle_0 r)(j) + r_1(j)]| \\
&\quad \le \frac{1}{2}\varepsilon\sum_{j=0}^{n}(\varphi\circ u)^2(j) + \frac{1}{2\varepsilon}\sum_{j=0}^{n}[q(\triangle_0 r)(j) + r_1(j)]^2. \qquad (4.25)
\end{aligned}
$$

Combining (4.25) and (4.24) and simplifying we obtain,

$$
\begin{aligned}
&q\Phi(u(n)) + \sum_{j=0}^{n}(\varphi\circ u)(j)u(j) - \frac{1}{a}\sum_{j=0}^{n}(\varphi\circ u)^2(j) + \frac{\varepsilon}{2}\sum_{j=0}^{n}(\varphi\circ u)^2(j) \\
&\quad + \frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)^2 \\
&\quad \le \frac{1}{2\varepsilon}\sum_{j=0}^{n}[q(\triangle_0 r)(j) + r_1(j)]^2 + \frac{1}{2}[\mathbf{G}(1)]^{-1}r_2{}^2. \qquad (4.26)
\end{aligned}
$$

The RHS of (4.26) can be estimated as follows,

$$
\begin{aligned}
&\frac{1}{2\varepsilon}\sum_{j=0}^{n}[q(\triangle_0 r)(j) + r_1(j)]^2 + \frac{1}{2}[\mathbf{G}(1)]^{-1}r_2{}^2 \\
&\le \frac{1}{2\varepsilon}\sum_{j=0}^{\infty}[q(\triangle_0 r)(j) + r_1(j)]^2 + \frac{1}{2}[\mathbf{G}(1)]^{-1}|r_2|^2 \\
&\le \frac{1}{\varepsilon}\sum_{j=0}^{\infty}[q^2(\triangle_0 r)^2(j) + r_1^2(j)] + \frac{1}{2}[\mathbf{G}(1)]^{-1}|r_2|^2 \\
&= \frac{q^2}{\varepsilon}\|\triangle_0 r\|_{l^2}^2 + \frac{1}{\varepsilon}\|r_1\|_{l^2}^2 + \frac{1}{2}[\mathbf{G}(1)]^{-1}|r_2|^2 \\
&\le \frac{q^2}{\varepsilon}(2\|r_1\|_{l^2} + |r_2|)^2 + \frac{1}{\varepsilon}\|r_1\|_{l^2}^2 + \frac{1}{2}[\mathbf{G}(1)]^{-1}|r_2|^2 \\
&\le \frac{8q^2}{\varepsilon}\|r_1\|_{l^2}^2 + \frac{2q^2}{\varepsilon}|r_2|^2 + \frac{1}{\varepsilon}\|r_1\|_{l^2}^2 + \frac{1}{2}[\mathbf{G}(1)]^{-1}|r_2|^2 \\
&\le L\|r\|_{m^2}^2, \qquad (4.27)
\end{aligned}
$$

where L depends only on q, ε and $\mathbf{G}(1)$ and we have used the fact that $\triangle_0 r = r_1 - \mathbf{S}r_1 + r_2\delta$ and so $\|\triangle_0 r\|_{l^2} \le 2\|r_1\|_{l^2} + |r_2|$. Combining (4.27) with (4.26) gives,

$$
\begin{aligned}
&q\Phi(u(n)) + \sum_{j=0}^{n}\left[(\varphi\circ u)(j)u(j) - \frac{1}{a}(\varphi\circ u)^2(j)\right] + \frac{\varepsilon}{2}\sum_{j=0}^{n}(\varphi\circ u)^2(j) \\
&+ \frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)^2 \le L\|r\|_{m^2}^2, \qquad \forall\, n \in \mathbb{Z}_+. \qquad (4.28)
\end{aligned}
$$

By sector condition (4.2) we have that,

$$\sum_{j=0}^{n}\left[(\varphi\circ u)(j)u(j)-\frac{1}{a}(\varphi\circ u)^2(j)\right]\geq 0.$$

We also note that, by (4.2), Φ is non-negative, as are all the other terms on the LHS of (4.28). Furthermore, the RHS of this inequality is entirely dependent on r and in particular does not depend on u. Inequality (4.28) is the key estimate from which we shall derive the theorem.

Proof of Statement 1: In the following, $K>0$ is a generic constant which will be suitably adjusted in every step and depends only on q, ε, a and G, but not on n, u or r. From (4.28) we obtain,

$$\frac{\varepsilon}{2}\sum_{j=0}^{n}(\varphi\circ u)^2(j)\leq L\|r\|_{m^2}^2,\qquad\forall\, n\in\mathbb{Z}_+.$$

Hence,

$$\|\varphi\circ u\|_{l^2}\leq K\|r\|_{m^2}. \tag{4.29}$$

Again, by (4.28),

$$\frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j)-[\mathbf{G}(1)]^{-1}r_2\right)^2\leq L\|r\|_{m^2}^2,\qquad\forall\, n\in\mathbb{Z}_+.$$

Hence,

$$\left|\left(\sum_{j=0}^{n}(\varphi\circ u)(j)-[\mathbf{G}(1)]^{-1}r_2\right)\right|^2\leq K\|r\|_{m^2}^2.$$

Now this implies that,

$$\left|\sum_{j=0}^{n}(\varphi\circ u)(j)\right|-|[\mathbf{G}(1)]^{-1}r_2|\leq K\|r\|_{m^2},\qquad\forall\, n\in\mathbb{Z}_+,$$

and so

$$\left|\sum_{j=0}^{n}(\varphi\circ u)(j)\right|\leq K\|r\|_{m^2},\qquad\forall\, n\in\mathbb{Z}_+.$$

Hence,

$$\sup_{n\geq 0}\left|\sum_{j=0}^{n}(\varphi\circ u)(j)\right|\leq K\|r\|_{m^2}. \tag{4.30}$$

Again starting with (4.28) we obtain the following inequality,

$$\sum_{j=0}^{n} \left[(\varphi \circ u)(j)u(j) - \frac{1}{a}(\varphi \circ u)^2(j) \right] \leq L\|r\|_{m^2}^2, \qquad \forall\, n \in \mathbb{Z}_+.$$

It follows that

$$\begin{aligned} 0 \leq \sum_{j=0}^{n} (\varphi \circ u)(j)u(j) &\leq L\|r\|_{m^2}^2 + \frac{1}{a}\sum_{j=0}^{\infty} (\varphi \circ u)^2(j) \\ &\leq K\|r\|_{m^2}^2, \qquad \forall\, n \in \mathbb{Z}_+, \end{aligned}$$

where we have used (4.29). Consequently,

$$(\|(\varphi \circ u)u\|_{l^1})^{1/2} \leq K\|r\|_{m^2}. \tag{4.31}$$

Since $\mathbf{H} \in H^\infty(\mathbb{E}_1) \subset H^2(\mathbb{E}_1)$, by Theorem 3.3.2, there exists $h \in l^2(\mathbb{Z}_+, \mathbb{R})$ such that,

$$\mathbf{H}(z) = \hat{h}(z) = \sum_{k=0}^{\infty} h(k) z^{-k}.$$

Hence,

$$\begin{aligned} |(Hv)(j)| = |\sum_{k=0}^{j} h(j-k)v(k)| &\leq \left(\sum_{k=0}^{j} |h(k)|^2 \right)^{1/2} \left(\sum_{k=0}^{j} |v(k)|^2 \right)^{1/2} \\ &\leq \|h\|_{l^2} \|v\|_{l^2}. \end{aligned} \tag{4.32}$$

Applying (4.32) with $v = \varphi \circ u$ and using (4.29) gives,

$$|(H(\varphi \circ u))(j)| \leq \|h\|_{l^2} \|\varphi \circ u\|_{l^2} \leq K\|r\|_{m^2}. \tag{4.33}$$

Invoking (4.8) we have,

$$\begin{aligned} |u(j)| &\leq |(H(\varphi \circ u))(j)| + |\mathbf{G}(1)||(J(\varphi \circ u))(j)| + |r(j)| \\ &\leq |(H(\varphi \circ u))(j)| + |\mathbf{G}(1)||(J(\varphi \circ u))(j)| + \|r\|_{m^2}. \end{aligned} \tag{4.34}$$

Taking the supremum and applying (4.30) combined with (4.33), we obtain from (4.34),

$$\|u\|_{l^\infty} = \sup_{n \geq 0} |u(n)| \leq K\|r\|_{m^2}. \tag{4.35}$$

By (4.7), we have that

$$\Delta_0 u = \Delta_0 r - \mathbf{S}G(\varphi \circ u). \tag{4.36}$$

Since $\varphi \circ u \in l^2(\mathbb{Z}_+, \mathbb{R})$ by (4.29), $G \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ and $\Delta_0 r \in l^2(\mathbb{Z}_+, \mathbb{R})$ we conclude that $\triangle_0 u \in l^2(\mathbb{Z}_+, \mathbb{R})$. Furthermore, by (4.29) the l^2-norm of the RHS of (4.36) is bounded by $K\|r\|_{m^2}$, and thus $\|\triangle_0 u\|_{l^2} \leq K\|r\|_{m^2}$. Combining this with (4.29), (4.30), (4.31) and (4.35) it is clear that (4.4) holds.

Proof of Statement 2: We note that by (4.13)

$$\begin{aligned}
&\frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi \circ u)(j)\right)^2 - \sum_{j=0}^{n}(\varphi \circ u)(j) r_2 \\
&\quad = \frac{\mathbf{G}(1)}{2}\sum_{j=0}^{n}(\varphi \circ u)^2(j) - q\sum_{j=0}^{n}(\varphi \circ u)(j)(\triangle_0 u)(j) \\
&\quad\quad - \sum_{j=0}^{n}(\varphi \circ u)(j)u(j) - \sum_{j=0}^{n}(\varphi \circ u)(j)(G_q(\varphi \circ u))(j) \\
&\quad\quad\quad + q\sum_{j=0}^{n}(\varphi \circ u)(j)(\triangle_0 r)(j) + \sum_{j=0}^{n}(\varphi \circ u)(j) r_1(j).
\end{aligned} \tag{4.37}$$

Completing the square on the LHS of (4.37) as in (4.23) we obtain,

$$\begin{aligned}
&\frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi \circ u)(j) - [\mathbf{G}(1)]^{-1} r_2\right)^2 \\
&= \frac{1}{2}[\mathbf{G}(1)]^{-1} {r_2}^2 - \sum_{j=0}^{n}(\varphi \circ u)(j)u(j) - \sum_{j=0}^{n}(\varphi \circ u)(j)(G_q(\varphi \circ u))(j) \\
&+ \sum_{j=0}^{n}(\varphi \circ u)(j) r_1(j) + \frac{\mathbf{G}(1)}{2}\sum_{j=0}^{n}(\varphi \circ u)^2(j) \\
&+ q\sum_{j=0}^{n}(\varphi \circ u)(j)(\triangle_0 r)(j) - q\sum_{j=0}^{n}(\varphi \circ u)(j)(\triangle_0 u)(j).
\end{aligned} \tag{4.38}$$

By statement 1, $\varphi \circ u \in l^2(\mathbb{Z}_+, \mathbb{R})$, $\triangle_0 u \in l^2(\mathbb{Z}_+, \mathbb{R})$ and $(\varphi \circ u)u \in l^1(\mathbb{Z}_+, \mathbb{R})$. Since $r_1 \in l^2(\mathbb{Z}_+, \mathbb{R})$ and $G_q \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ it follows that, $(\varphi \circ u)G_q(\varphi \circ u)$, $(\varphi \circ u)r_1$, $(\varphi \circ u)^2$, $(\varphi \circ u)(\triangle_0 r)$ and $(\varphi \circ u)(\triangle_0 u)$ are in $l^1(\mathbb{Z}_+, \mathbb{R})$, and the RHS of (4.38) has a finite limit as $n \to \infty$. Hence,

$$\lambda := \lim_{n \to \infty}\left(\sum_{j=0}^{n}(\varphi \circ u)(j) - [\mathbf{G}(1)]^{-1} r_2\right)^2$$

exists and is finite. The problem now is that as $n \to \infty$, $\sum_{j=0}^{n}(\varphi \circ u)(j)$ could jump between small neighbourhoods of the points $[\mathbf{G}(1)]^{-1} r_2 \pm \sqrt{\lambda}$ and not converge to either of them. To see that this is not the case, we set $w(n) := \sum_{j=0}^{n}(\varphi \circ u)(j)$.

By statement 1, $(\varphi \circ u)(j) \to 0$ as $j \to \infty$ since $\varphi \circ u \in l^2(\mathbb{Z}_+, \mathbb{R})$. Hence, $(\Delta w)(n) = (\varphi \circ u)(n+1) \to 0$ and we see that $w(n) = \sum_{j=0}^{n} (\varphi \circ u)(j)$ must converge to one of the points $[\mathbf{G}(1)]^{-1} r_2 \pm \sqrt{\lambda}$. To prove that $\lim_{n\to\infty} u(n)$ exists, it is sufficient to show that,

$$\lim_{n\to\infty} \Big(u(n) + \mathbf{G}(1)(J(\varphi \circ u))(n) \Big) = r_2. \tag{4.39}$$

By (4.8), (4.39) is equivalent to the claim that $\lim_{n\to\infty}(H(\varphi \circ u))(n) = 0$. The later follows from the fact that, $H \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ and $\varphi \circ u \in l^2(\mathbb{Z}_+, \mathbb{R})$ by (4.4), showing that $H(\varphi \circ u) \in l^2(\mathbb{Z}_+, \mathbb{R})$. Consequently, $H(\varphi \circ u)(n) \to 0$ as $n \to \infty$, implying that (4.39) holds. Therefore, $u^\infty := \lim_{n\to\infty} u(n)$ exists and is finite. Finally, noting that by statement 1, $\varphi \circ u \in l^2(\mathbb{Z}_+, \mathbb{R})$ and assuming that φ is continuous, it is clear that $\varphi(u^\infty) = 0$. □

We now assume that $\varepsilon = 0$ in the positive-real condition (4.3) and consider the absolute stability problem shown in Figure 4.1.

Theorem 4.1.3. *Let $G \in \mathscr{B}(l^\infty(\mathbb{Z}_+, \mathbb{R}))$ be a shift-invariant operator with transfer function $\mathbf{G}$ satisfying assumption* (A) *and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a measurable non-decreasing non-linearity. Assume that $\mathbf{G}(1) > 0$ and there exists numbers $q \geq 0$ and $a \in (0, \infty)$ such that* (4.2) *holds and*

$$\frac{1}{a} + \operatorname{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta} - 1}\right)\mathbf{G}(e^{i\theta})\right] \geq 0, \qquad \text{a.a. } \theta \in (0, 2\pi). \tag{4.40}$$

Let $r \in m^1(\mathbb{Z}_+, \mathbb{R})$, and let $u : \mathbb{Z}_+ \to \mathbb{R}$ be the unique solution of (4.1). *Then there exists a constant $K > 0$ (which depends only on q and G, but not on r) such that,*

$$\|u\|_{l^\infty} + \sup_{n\geq 0}\left|\sum_{j=0}^{n} (\varphi \circ u)(j)\right| + (\|(\varphi \circ u)(u - \frac{1}{a}\varphi \circ u)\|_{l^1})^{1/2} \leq K \|r\|_{m^1}. \tag{4.41}$$

Remark 4.1.4. We claim that under the assumptions of Theorem 4.1.3, (4.40) can not be satisfied for $a = \infty$. To this end consider the function $\mathbf{G}_q$ defined by,

$$\mathbf{G}_q(z) = \frac{q}{z}\mathbf{G}(z) + \frac{1}{z-1}[\mathbf{G}(z) - \mathbf{G}(1)], \qquad z \in \mathbb{E}_1,$$

which has been used in the proof of Theorem 4.1.2. From assumption (A) and the fact that $\mathbf{G} \in H^\infty(\mathbb{E}_1)$, we have that $\mathbf{G}_q \in H^\infty(\mathbb{E}_1)$. Since

$$\lim_{|z|\to\infty, z\in\mathbb{E}_1} \mathbf{G}_q(z) = 0,$$

we see that by the property of $z \mapsto e^{-\mathbf{G}_q(z)} \in H^\infty(\mathbb{E}_1)$,

$$1 \le \sup_{z\in\mathbf{E}_1} |e^{-\mathbf{G}_q(z)}| = \operatorname{ess\,sup}_{\theta\in(0,2\pi)} |e^{-\mathbf{G}_q(e^{i\theta})}| = e^{-\widetilde{G}_q},$$

where $\widetilde{G}_q := \operatorname{ess\,inf}_{\theta\in(0,2\pi)} \mathrm{Re}\,\mathbf{G}_q(e^{i\theta})$. Hence $\widetilde{G}_q \le 0$ and, consequently,

$$\begin{aligned}
&\operatorname{ess\,inf}_{\theta\in(0,2\pi)} \mathrm{Re}\,\mathbf{G}_q(e^{i\theta}) \\
&\quad = \operatorname{ess\,inf}_{\theta\in(0,2\pi)} \mathrm{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta}) - \frac{1}{e^{i\theta}-1}\mathbf{G}(1)\right] \\
&\quad = \operatorname{ess\,inf}_{\theta\in(0,2\pi)} \mathrm{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta}-1}\right)\right] + \frac{\mathbf{G}(1)}{2} \le 0.
\end{aligned} \tag{4.42}$$

Since $\mathbf{G}(1) > 0$, it follows from (4.42) that

$$\operatorname{ess\,inf}_{\theta\in(0,2\pi)} \mathrm{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right] \le -\frac{\mathbf{G}(1)}{2}.$$

This implies that if (4.40) holds, then $a \le 2/\mathbf{G}(1) < \infty$. $\diamond$

Proof of Theorem 4.1.3. Defining,

$$\sigma(n) := \sum_{j=0}^{n} (\varphi \circ u)(j), \quad \forall\, n \in \mathbb{Z}_+, \quad \sigma(-1) := 0,$$

and setting $\varepsilon = 0$ in (4.20) we obtain,

$$\begin{aligned}
&q\Phi(u(n)) + \sum_{j=0}^{n}\left[(\varphi\circ u)(j)(u(j) - \frac{1}{a}(\varphi\circ u)(j))\right] + \frac{\mathbf{G}(1)}{2}\sigma^2(n) \\
&\qquad \le q\sum_{j=0}^{n}(\varphi\circ u)(j)(\Delta_0 r)(j) + \sum_{j=0}^{n}(\varphi\circ u)(j) r(j),
\end{aligned} \tag{4.43}$$

where $\Phi(v) := \int_0^v \varphi(\sigma)\,d\sigma$ for all $v \in \mathbb{R}$. By assumption $r = r_1 + r_2\vartheta$, where $r_1 \in l^1(\mathbb{Z}_+, \mathbb{R})$ and $r_2 \in \mathbb{R}$. Using this decomposition of r we can write the RHS of (4.43) as,

$$\begin{aligned}
&q\sum_{j=0}^{n}(\varphi\circ u)(j)(\Delta_0 r)(j) + \sum_{j=0}^{n}(\varphi\circ u)(j) r(j) \\
&= \sum_{j=0}^{n}[\sigma(j) - \sigma(j-1)][q(\Delta_0 r)(j) + r_1(j)] + \sigma(n) r_2.
\end{aligned} \tag{4.44}$$

Partial summation yields the identities,

$$\sum_{j=0}^{n}[\sigma(j)-\sigma(j-1)](\triangle_0 r)(j)=\sum_{j=0}^{n}\sigma(j)((\triangle_0 r)(j)-(\triangle_0 r)(j+1)) \\ +\sigma(n)(\triangle_0 r)(n+1) \tag{4.45}$$

and

$$\sum_{j=0}^{n}[\sigma(j)-\sigma(j-1)]r_1(j)=\sum_{j=0}^{n}\sigma(j)[r_1(j)-r_1(j+1)] \\ +\sigma(n)r_1(n+1). \tag{4.46}$$

Combining (4.46) and (4.45) with (4.44) we obtain, from (4.43),

$$\begin{aligned} &q\Phi(u(n))+\sum_{j=0}^{n}\left[(\varphi\circ u)(j)(u(j)-\frac{1}{a}(\varphi\circ u)(j))\right]+\frac{\mathbf{G}(1)}{2}\sigma(n)^2 \\ &\leq q\sum_{j=0}^{n}\sigma(j)((\triangle_0 r)(j)-(\triangle_0 r)(j+1))+q\sigma(n)(\triangle_0 r)(n+1) \\ &\quad+\sum_{j=0}^{n}\sigma(j)[r_1(j)-r_1(j+1)]+\sigma(n)r_1(n+1)+\sigma(n)r_2. \end{aligned} \tag{4.47}$$

The RHS of (4.47) can be estimated as follows,

$$\begin{aligned} &q\sum_{j=0}^{n}[\sigma(j)((\triangle_0 r)(j)-(\triangle_0 r)(j+1))]+q\sigma(n)(\triangle_0 r)(n+1) \\ &+\sum_{j=0}^{n}\sigma(j)[r_1(j)-r_1(j+1)]+\sigma(n)r_1(n+1)+\sigma(n)r_2 \\ &\leq 2q\|\triangle_0 r\|_{l^1}\max_{0\leq j\leq n}|\sigma(j)|+q|\sigma(n)||\triangle_0 r(n+1)|+2\|r_1\|_{l^1}\max_{0\leq j\leq n}|\sigma(j)| \\ &\qquad+|\sigma(n)||r_1(n+1)|+|\sigma(n)||r_2| \\ &\leq \max_{0\leq j\leq n}|\sigma(j)|(3q\|\triangle_0 r\|_{l^1}+3\|r_1\|_{l^1}+|r_2|) \\ &\leq \max_{0\leq j\leq n}|\sigma(j)|(3q(2\|r_1\|_{l^1}+|r_2|)+3\|r_1\|_{l^1}+|r_2|) \\ &\leq L\max_{0\leq j\leq n}|\sigma(j)|\|r\|_{m^1}, \end{aligned} \tag{4.48}$$

where $L>0$ is some constant depending only on q. Combining (4.48) with (4.47) gives,

$$q\Phi(u(n)) + \sum_{j=0}^{n}\left[(\varphi\circ u)(j)(u(j) - \frac{1}{a}(\varphi\circ u)(j))\right] + \frac{\mathbf{G}(1)}{2}\sigma(n)^2$$
$$\leq L\max_{0\leq j\leq n}|\sigma(j)|\|r\|_{m^1}. \qquad (4.49)$$

In the following, $K > 0$ is a generic constant which will be suitably adjusted in every step and depends only on q and G, but not on u, r or n.

Since Φ is non-negative, the first term on the LHS of this equation is non-negative and by sector condition (4.2), the second term on the LHS of (4.49) is also non-negative. Hence, from (4.49) we obtain,

$$|\sigma(n)| = \left|\sum_{j=0}^{n}(\varphi\circ u)(j)\right| \leq K\|r\|_{m^1}, \quad \forall\, n\in\mathbb{Z}_+,$$

which in turn implies that

$$\|\sigma\|_{l^\infty} = \sup_{n\geq 0}\left|\sum_{j=0}^{n}(\varphi\circ u)(j)\right| \leq K\|r\|_{m^1}. \qquad (4.50)$$

Using (4.50) in (4.49) shows that,

$$(\|(\varphi\circ u)(u - \frac{1}{a}(\varphi\circ u))\|_{l^1})^{1/2} \leq K\|r\|_{m^1}. \qquad (4.51)$$

Since G is shift-invariant, it commutes with J, and hence, from (4.6),

$$\begin{aligned}|u(n)| &= |r(n) - (J(G(\varphi\circ u)))(n)|\\ &\leq |r(n)| + |(G(J(\varphi\circ u)))(n)|\\ &\leq \|r\|_{m^1} + \|G\|\|\sigma\|_{l^\infty}.\end{aligned}$$

Taking the supremum and applying (4.50) we obtain,

$$\|u\|_{l^\infty} = \sup_{n\geq 0}|u(n)| \leq K\|r\|_{m^1}.$$

Together with (4.51) and (4.50) it now follows that (4.41) holds. □

4.2 Absolute stability results of circle-criterion-type

We now consider the absolute stability problem shown in Figure 4.1 where φ is time-dependent. Recall that U denotes a (possibly complex) Hilbert space.

Theorem 4.2.1. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ be a shift-invariant operator with transfer function $\mathbf{G}$ satisfying assumption* (A) *with* $\mathbf{G}(1)$ *invertible, and let $\varphi : \mathbb{Z}_+ \times U \to U$ be a time-varying non-linearity. Assume that there exist self-adjoint $P \in \mathscr{B}(U)$, invertible $Q \in \mathscr{B}(U)$ with $Q\mathbf{G}(1) = [Q\mathbf{G}(1)]^* \geq 0$ and a number $\varepsilon > 0$ such that*

$$\operatorname{Re}\langle \varphi(n,v), Qv \rangle \geq \langle \varphi(n,v), P\varphi(n,v) \rangle, \qquad \forall\, n \in \mathbb{Z}_+,\ v \in U, \tag{4.52}$$

and

$$P + \frac{1}{2}\left[\frac{1}{e^{i\theta}-1} Q\mathbf{G}(e^{i\theta}) + \frac{1}{e^{-i\theta}-1}\mathbf{G}^*(e^{i\theta})Q^*\right] \geq \varepsilon I, \qquad \text{a.a. } \theta \in (0, 2\pi). \tag{4.53}$$

Let $r \in m^2(\mathbb{Z}_+, U)$, that is, $r = r_1 + r_2\vartheta$ with $r_1 \in l^2(\mathbb{Z}_+, U)$ and $r_2 \in U$, and let $u : \mathbb{Z}_+ \to U$ be the unique solution of (4.1). *Then the following statements hold.*

1. *There exists a constant K (which depends only on ε, P, Q and G, but not on r) such that,*

$$\|u\|_{l^\infty} + \|\Delta u\|_{l^2} + \|\varphi \circ u\|_{l^2} + (\|\operatorname{Re}\langle(\varphi \circ u), Qu\rangle\|_{l^1})^{1/2} + \sup_{n\geq 0}\left\|\sum_{j=0}^{n}(\varphi \circ u)(j)\right\| \leq K\|r\|_{m^2}. \tag{4.54}$$

2. *We have,*

$$\lim_{n\to\infty}\left(u(n) + \mathbf{G}(1)\sum_{j=0}^{n}(\varphi \circ u)(j)\right) = r_2; \tag{4.55}$$

in particular, $\lim_{n\to\infty} u(n)$ *exists if and only if* $\lim_{n\to\infty}\sum_{j=0}^{n}(\varphi \circ u)(j)$ *exists, in which case*

$$\lim_{n\to\infty} u(n) = r_2 - \mathbf{G}(1)\lim_{n\to\infty}\sum_{j=0}^{n}(\varphi \circ u)(j). \tag{4.56}$$

3. *There exists a sphere $S \subset U$ centred at* 0, *such that*

$$\lim_{n\to\infty}\operatorname{dist}(u(n), \mathbf{G}(1)[Q\mathbf{G}(1)]^{-1/2}S) = 0, \tag{4.57}$$

in particular, if $\dim U = 1$*, then* $\lim_{n\to\infty} u(n)$ *and* $\lim_{n\to\infty} \sum_{j=0}^{n} (\varphi \circ u)(j)$ *exist.*

4. *If we relax condition* (4.52) *and only require that for some* $n_0 > 0$*,*

$$\operatorname{Re} \langle \varphi(n,v), Qv \rangle \geq \langle \varphi(n,v), P\varphi(n,v) \rangle , \qquad \forall\, n \geq n_0, v \in U, \qquad (4.58)$$

then the LHS of (4.54) *is still finite (but no longer bounded in terms of* $\|r\|_{m^2}$*) and statements 2 and 3 remain valid.*

5. *Under the additional assumptions*

   (B) φ *does not depend on time,*

   (C) $\varphi^{-1}(0) \cap B$ *is precompact for every bounded set* $B \subset U$*,*

   (D) $\inf_{v\in B} \|\varphi(v)\| > 0$ *for every bounded, closed set* $B \subset U$ *such that* $\varphi^{-1}(0) \cap B = \emptyset$*,*

   we have that $\lim_{n\to\infty} \operatorname{dist}(u(n), \varphi^{-1}(0)) = 0$*.*

6. *If the additional assumptions* (B)-(D) *of statement 5 hold and if the intersection* $\operatorname{cl}(\varphi^{-1}(0)) \cap \mathbf{G}(1)[Q\mathbf{G}(1)]^{-1/2} S$ *is totally disconnected for every sphere* $S \subset U$ *centred at* 0*, then* $u(n)$ *converges as* $n \to \infty$*.*

Remark 4.2.2. Note that if $\dim U < \infty$ then assumption (C) always holds. If $\dim U < \infty$, φ is time-independent and φ is continuous, then assumption (D) is satisfied. ◇

Before proving Theorem 4.2.1, we state a slightly simplified version of this result (where P and Q are scalars and $P \geq 0$) in the form of a corollary which is convenient in the context of applications of Theorem 4.2.1 to integral control (see Chapter 5).

Corollary 4.2.3. *Let* $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ *be a shift-invariant operator with transfer function* $\mathbf{G}$ *satisfying assumption* (A) *with* $\mathbf{G}(1)$ *invertible,* $\mathbf{G}(1) = \mathbf{G}^*(1) \geq 0$*, and let* $\varphi : \mathbb{Z}_+ \times U \to U$ *be a time-varying non-linearity. Assume that there exists numbers* $\varepsilon > 0$ *and* $a \in (0, \infty)$ *such that*

$$\operatorname{Re} \langle \varphi(n,v), v \rangle \geq \frac{1}{a} \|\varphi(n,v)\|^2, \qquad \forall\, v \in U, \qquad (4.59)$$

and

$$\frac{1}{a} I + \frac{1}{2} \left[\frac{1}{e^{i\theta} - 1} \mathbf{G}(e^{i\theta}) + \frac{1}{e^{-i\theta} - 1} \mathbf{G}^*(e^{i\theta}) \right] \geq \varepsilon I, \qquad \text{a.a. } \theta \in (0, 2\pi).$$

Then for all $r \in m^2(\mathbb{Z}_+, U)$*, the conclusions of Theorem* 4.2.1 *hold with* $P = (1/a)I$ *and* $Q = I$*.*

Proof of Theorem 4.2.1. We have,

$$u + (JG)(\varphi \circ u) = r. \tag{4.60}$$

We now write (4.60) in a slightly more convenient form, namely,

$$u + H(\varphi \circ u) + \mathbf{G}(1)J(\varphi \circ u) = r, \tag{4.61}$$

where $H := J \circ G - \mathbf{G}(1)J$. It is clear that this operator is shift-invariant with transfer function $\mathbf{H}$ given by,

$$\mathbf{H}(z) = \frac{1}{z-1}[\mathbf{G}(z) - \mathbf{G}(1)], \qquad z \in \mathbb{E}_1. \tag{4.62}$$

From assumption (A) and from the fact that $\mathbf{G} \in H^\infty(\mathbb{E}_1, \mathscr{B}(U))$, we conclude that $\mathbf{H} \in H^\infty(\mathbb{E}_1, \mathscr{B}(U))$, and hence $H \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$.

Applying Q to (4.61) we obtain,

$$Qu(j) + (G_Q(\varphi \circ u))(j) + Q\mathbf{G}(1)(J(\varphi \circ u))(j) = Qr(j), \qquad \forall\, j \in \mathbb{Z}_+, \tag{4.63}$$

where we have defined the operator G_Q by $G_Q := QH$. Invoking (4.62) we see that the transfer function $\mathbf{G}_Q$ of G_Q is given by

$$\mathbf{G}_Q := \frac{1}{z-1} Q[\mathbf{G}(z) - \mathbf{G}(1)], \qquad z \in \mathbb{E}_1.$$

Forming the inner product with $(\varphi \circ u)(j)$, taking real parts and summing from 0 to n in (4.63) yields,

$$\begin{aligned}
\mathrm{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qu(j) \rangle &+ \mathrm{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), (G_Q(\varphi \circ u))(j) \rangle \\
&+ \mathrm{Re} \sum_{j=1}^{n} \langle (\varphi \circ u)(j), Q\mathbf{G}(1) \sum_{k=0}^{j-1} (\varphi \circ u)(k) \rangle \\
&= \mathrm{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qr(j) \rangle.
\end{aligned} \tag{4.64}$$

Since, by assumption, $Q\mathbf{G}(1) = [Q\mathbf{G}(1)]^* \geq 0$, the square root $[Q\mathbf{G}(1)]^{1/2}$ of $Q\mathbf{G}(1)$ exists and hence, an application of Lemma 4.1.1 to the last term on the

LHS of (4.64) yields,

$$
\begin{aligned}
&\operatorname{Re}\sum_{j=1}^{n}\langle(\varphi\circ u)(j), Q\mathbf{G}(1)\sum_{k=0}^{j-1}(\varphi\circ u)(k)\rangle \\
&= \operatorname{Re}\sum_{j=1}^{n}\langle[Q\mathbf{G}(1)]^{1/2}(\varphi\circ u)(j), \sum_{k=0}^{j-1}[Q\mathbf{G}(1)]^{1/2}(\varphi\circ u)(k)\rangle \\
&= \frac{1}{2}\left\|[Q\mathbf{G}(1)]^{1/2}\sum_{j=0}^{n}(\varphi\circ u)(j)\right\|^2 - \frac{1}{2}\sum_{j=0}^{n}\|[Q\mathbf{G}(1)]^{1/2}(\varphi\circ u)(j)\|^2. \qquad (4.65)
\end{aligned}
$$

Combining (4.65) with (4.64) gives,

$$
\begin{aligned}
\operatorname{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j), Qu(j)\rangle &+ \operatorname{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j), (G_Q(\varphi\circ u))(j)\rangle \\
&+ \frac{1}{2}\left\|[Q\mathbf{G}(1)]^{1/2}\sum_{j=0}^{n}(\varphi\circ u)(j)\right\|^2 \\
&- \frac{1}{2}\sum_{j=0}^{n}\|[Q\mathbf{G}(1)]^{1/2}(\varphi\circ u)(j)\|^2 \\
&= \operatorname{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j), Qr(j)\rangle. \qquad (4.66)
\end{aligned}
$$

We note that,

$$
\begin{aligned}
\mathbf{G}_Q(z) + \mathbf{G}_Q^*(z) &= \frac{1}{z-1}Q[\mathbf{G}(z) - \mathbf{G}(1)] + \left(\frac{1}{z-1}Q[\mathbf{G}(z) - \mathbf{G}(1)]\right)^* \\
&= \frac{1}{z-1}Q\mathbf{G}(z) + \frac{1}{\bar{z}-1}\mathbf{G}^*(z)Q^* - \left(\frac{1}{z-1} + \frac{1}{\bar{z}-1}\right)Q\mathbf{G}(1).
\end{aligned}
$$

Taking $z = e^{i\theta}$ with $\theta \in (0, 2\pi)$ yields,

$$
\frac{1}{e^{i\theta}-1} + \frac{1}{e^{-i\theta}-1} = \frac{e^{-i\theta}-1+e^{i\theta}-1}{(e^{i\theta}-1)(e^{-i\theta}-1)} = \frac{e^{-i\theta}+e^{i\theta}-2}{2-e^{i\theta}-e^{-i\theta}} = -1,
$$

and we obtain,

$$
\begin{aligned}
\mathbf{G}_Q(e^{i\theta}) + \mathbf{G}_Q^*(e^{i\theta}) = \frac{1}{e^{i\theta}-1}Q\mathbf{G}(e^{i\theta}) &+ \frac{1}{e^{-i\theta}-1}\mathbf{G}^*(e^{i\theta})Q^* \\
&+ Q\mathbf{G}(1), \quad \text{a.a. } \theta \in (0, 2\pi). \qquad (4.67)
\end{aligned}
$$

Combining (4.67) with (4.53) we see that,

$$\begin{aligned}\frac{1}{2}[\mathbf{G}_Q(e^{i\theta})+\mathbf{G}_Q^*(e^{i\theta})] &= \frac{1}{2}\left[\frac{1}{e^{i\theta}-1}Q\mathbf{G}(e^{i\theta})+\frac{1}{e^{-i\theta}-1}\mathbf{G}^*(e^{i\theta})Q^*\right]+\frac{Q\mathbf{G}(1)}{2}\\ &\geq \varepsilon I - P + \frac{Q\mathbf{G}(1)}{2}. \end{aligned} \tag{4.68}$$

Define $v:\mathbb{Z}\to U$ by

$$v(j)=\begin{cases}(\varphi\circ u)(j), & \text{if } 0\leq j\leq n,\\ 0, & \text{otherwise.}\end{cases} \tag{4.69}$$

By Theorem 3.3.3,

$$\begin{aligned}\operatorname{Re}\sum_{j=0}^{\infty}\langle v(j),(G_Qv)(j)\rangle &= \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\langle\hat{v}(e^{i\theta}),(\widehat{G_Qv})(e^{i\theta})\rangle\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\langle\hat{v}(e^{i\theta}),\mathbf{G}_Q(e^{i\theta})\hat{v}(e^{i\theta})\rangle\,d\theta. \end{aligned}\tag{4.70}$$

Using (4.68), noting that $Q\mathbf{G}(1)=[Q\mathbf{G}(1)]^*\geq 0$ and applying again Theorem 3.3.3, we obtain,

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\langle\hat{v}(e^{i\theta}),\mathbf{G}_Q(e^{i\theta})\hat{v}(e^{i\theta})\rangle\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}\left\langle\hat{v}(e^{i\theta}),\frac{1}{2}\Big(\mathbf{G}_Q(e^{i\theta})+\mathbf{G}_Q^*(e^{i\theta})\Big)\hat{v}(e^{i\theta})\right\rangle d\theta\\ &\geq \frac{1}{2\pi}\int_0^{2\pi}\left\langle\hat{v}(e^{i\theta}),\left(\varepsilon I-P+\frac{Q\mathbf{G}(1)}{2}\right)\hat{v}(e^{i\theta})\right\rangle d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}\varepsilon\|\hat{v}(e^{i\theta})\|^2\,d\theta-\frac{1}{2\pi}\int_0^{2\pi}\langle\hat{v}(e^{i\theta}),P\hat{v}(e^{i\theta})\rangle\,d\theta\\ &\qquad+\frac{1}{4\pi}\int_0^{2\pi}\|[Q\mathbf{G}(1)]^{1/2}\hat{v}(e^{i\theta})\|^2\,d\theta\\ &= \varepsilon\sum_{j=0}^{\infty}\|v(j)\|^2-\sum_{j=0}^{\infty}\langle v(j),Pv(j)\rangle+\frac{1}{2}\sum_{j=0}^{\infty}\|[Q\mathbf{G}(1)]^{1/2}v(j)\|^2.\end{aligned}$$

Invoking (4.69) and (4.70), we conclude that,

$$\begin{aligned}
&\operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), (G_Q(\varphi \circ u))(j) \rangle \\
&\qquad \geq \varepsilon \sum_{j=0}^{n} \|(\varphi \circ u)(j)\|^2 - \sum_{j=0}^{n} \langle (\varphi \circ u)(j), P(\varphi \circ u)(j) \rangle \\
&\qquad\qquad + \frac{1}{2} \sum_{j=0}^{n} \|[Q\mathbf{G}(1)]^{1/2} (\varphi \circ u)(j)\|^2.
\end{aligned} \tag{4.71}$$

Applying (4.71) to (4.66) gives,

$$\begin{aligned}
&\operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qu(j) \rangle - \sum_{j=0}^{n} \langle (\varphi \circ u)(j), P(\varphi \circ u)(j) \rangle + \varepsilon \sum_{j=0}^{n} \|(\varphi \circ u)(j)\|^2 \\
&\quad + \frac{1}{2} \left\| [Q\mathbf{G}(1)]^{1/2} \left(\sum_{j=0}^{n} (\varphi \circ u)(j) \right) \right\|^2 \leq \operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qr(j) \rangle.
\end{aligned} \tag{4.72}$$

By assumption $r = r_1 + r_2 \vartheta$, where $r_1 \in l^2(\mathbb{Z}_+, U)$ and $r_2 \in U$. Using this decomposition of r we can write the term on the RHS of (4.72) as,

$$\operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qr(j) \rangle = \operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qr_1(j) \rangle + \operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qr_2 \rangle. \tag{4.73}$$

Using (4.73) in (4.72) gives,

$$\begin{aligned}
&\operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qu(j) \rangle - \sum_{j=0}^{n} \langle (\varphi \circ u)(j), P(\varphi \circ u)(j) \rangle + \varepsilon \sum_{j=0}^{n} \|(\varphi \circ u)(j)\|^2 \\
&\qquad + \frac{1}{2} \left\| [Q\mathbf{G}(1)]^{1/2} \left(\sum_{j=0}^{n} (\varphi \circ u)(j) \right) \right\|^2 \\
&\qquad \leq \operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qr_1(j) \rangle + \operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qr_2 \rangle.
\end{aligned} \tag{4.74}$$

Completing the square on the expression,

$$\frac{1}{2} \left\| [Q\mathbf{G}(1)]^{1/2} \sum_{j=0}^{n} (\varphi \circ u)(j) \right\|^2 - \operatorname{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qr_2 \rangle,$$

we obtain,

$$
\begin{aligned}
&\frac{1}{2}\left\|[Q\mathbf{G}(1)]^{1/2}\sum_{j=0}^{n}(\varphi\circ u)(j)\right\|^2 - \mathrm{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j), Qr_2\rangle \\
&= \frac{1}{2}\left\|[Q\mathbf{G}(1)]^{1/2}\sum_{j=0}^{n}(\varphi\circ u)(j)\right\|^2 \\
&\quad -\mathrm{Re}\left\langle [Q\mathbf{G}(1)]^{1/2}\sum_{j=0}^{n}(\varphi\circ u)(j), [Q\mathbf{G}(1)]^{1/2}[\mathbf{G}(1)]^{-1}r_2\right\rangle \\
&= \frac{1}{2}\left\|[Q\mathbf{G}(1)]^{1/2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)\right\|^2 \\
&\quad -\frac{1}{2}\|[Q\mathbf{G}(1)]^{1/2}[\mathbf{G}(1)]^{-1}r_2\|^2.
\end{aligned}
\tag{4.75}
$$

So using (4.75) in (4.74) yields the following,

$$
\begin{aligned}
&\mathrm{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j), Qu(j)\rangle - \sum_{j=0}^{n}\langle(\varphi\circ u)(j), P(\varphi\circ u)(j)\rangle + \varepsilon\sum_{j=0}^{n}\|(\varphi\circ u)(j)\|^2 \\
&+\frac{1}{2}\left\|[Q\mathbf{G}(1)]^{1/2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)\right\|^2 \\
&\qquad \le \mathrm{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j), Qr_1(j)\rangle + \frac{1}{2}\|[Q\mathbf{G}(1)]^{1/2}[\mathbf{G}(1)]^{-1}r_2\|^2.
\end{aligned}
\tag{4.76}
$$

Now the first term on the RHS of (4.76) can be estimated as follows (since $ab \le \frac{1}{2}\varepsilon a^2 + b^2/(2\varepsilon)$ for non-negative numbers a and b)

$$
\begin{aligned}
\mathrm{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j), Qr_1(j)\rangle &\le \sum_{j=0}^{n}\|(\varphi\circ u)(j)\|\|Qr_1(j)\| \\
&\le \frac{1}{2}\varepsilon\sum_{j=0}^{n}\|(\varphi\circ u)(j)\|^2 + \frac{1}{2\varepsilon}\sum_{j=0}^{n}\|Qr_1(j)\|^2.
\end{aligned}
\tag{4.77}
$$

Combining (4.77) and (4.76) and simplifying we obtain,

$$\begin{aligned}
\operatorname{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j), Qu(j)-P(\varphi\circ u)(j)\rangle + \frac{\varepsilon}{2}\sum_{j=0}^{n}\|(\varphi\circ u)(j)\|^2 \\
+\frac{1}{2}\left\|[Q\mathbf{G}(1)]^{1/2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)\right\|^2 \\
\leq \frac{1}{2\varepsilon}\sum_{j=0}^{\infty}\|Qr_1(j)\|^2 + \frac{1}{2}\|[Q\mathbf{G}(1)]^{1/2}[\mathbf{G}(1)]^{-1}r_2\|^2 \\
\leq \frac{\|Q\|^2}{2\varepsilon}\|r_1\|_{l^2}^2 + \frac{1}{2}\|[Q\mathbf{G}(1)]^{1/2}[\mathbf{G}(1)]^{-1}\|^2\|r_2\|^2 \\
\leq L\|r\|_{m^2}^2, \qquad \forall\, n\in\mathbb{Z}_+,
\end{aligned} \tag{4.78}$$

where L depends only on ε, G and Q. By the sector condition (4.52) we have that,

$$\operatorname{Re}\langle(\varphi\circ u)(j), Qu(j)-P(\varphi\circ u)(j)\rangle \geq 0, \qquad \forall\, j\in\mathbb{Z}_+,$$

showing that the first term on the LHS of (4.78) is non-negative. We also note that all the other terms on the LHS of (4.78) are non-negative. Inequality (4.78) is the key estimate from which we will derive the theorem.

Proof of Statement 1: In the following, $K>0$ is a generic constant which will be suitably adjusted in every step and depends only on ε, P, Q and G, but not on n or r. From (4.78) we obtain,

$$\frac{\varepsilon}{2}\sum_{j=0}^{n}\|(\varphi\circ u)(j)\|^2 \leq L\|r\|_{m^2}^2, \qquad \forall\, n\in\mathbb{Z}_+.$$

Hence,

$$\|\varphi\circ u\|_{l^2} \leq K\|r\|_{m^2}. \tag{4.79}$$

Again, by (4.78),

$$\frac{1}{2}\left\|[Q\mathbf{G}(1)]^{1/2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)\right\|^2 \leq L\|r\|_{m^2}^2, \qquad \forall\, n\in\mathbb{Z}_+.$$

Since $Q\mathbf{G}(1)$ is invertible (by assumption), $[Q\mathbf{G}(1)]^{1/2}$ is invertible and therefore,

$$\left\|\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)\right\|^2 \leq K\|r\|_{m^2}^2.$$

Now this implies that,

$$\left\|\sum_{j=0}^{n}(\varphi\circ u)(j)\right\| - \|[\mathbf{G}(1)]^{-1}r_2\| \le K\|r\|_{m^2}, \qquad \forall\, n\in\mathbb{Z}_+,$$

and so,

$$\left\|\sum_{j=0}^{n}(\varphi\circ u)(j)\right\| \le K\|r\|_{m^2}, \qquad \forall\, n\in\mathbb{Z}_+.$$

Hence,

$$\sup_{n\ge 0}\left\|\sum_{j=0}^{n}(\varphi\circ u)(j)\right\| \le K\|r\|_{m^2}. \tag{4.80}$$

Again starting with (4.78), we obtain the following inequality,

$$\operatorname{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j), Qu(j) - P(\varphi\circ u)(j)\rangle \le L\|r\|_{m^2}^2, \qquad \forall\, n\in\mathbb{Z}_+.$$

Since,

$$\begin{aligned}|\operatorname{Re}\langle(\varphi\circ u)(j), Qu(j)\rangle| &\le |\operatorname{Re}\langle(\varphi\circ u)(j), Qu(j) - P(\varphi\circ u)(j)\rangle| \\ &\quad + |\operatorname{Re}\langle(\varphi\circ u)(j), P(\varphi\circ u)(j)\rangle|\end{aligned}$$

it follows that,

$$\begin{aligned}\sum_{j=0}^{n}|\operatorname{Re}\langle(\varphi\circ u)(j), Qu(j)\rangle| &\le L\|r\|_{m^2}^2 + \|P\|\sum_{j=0}^{\infty}\|(\varphi\circ u)^2(j)\| \\ &\le K\|r\|_{m^2}^2, \qquad \forall\, n\in\mathbb{Z}_+,\end{aligned}$$

where we have used (4.79). Consequently,

$$(\|\operatorname{Re}\langle(\varphi\circ u), Qu\rangle\|_{l^1})^{1/2} \le K\|r\|_{m^2}. \tag{4.81}$$

Since $H\in\mathscr{B}(l^2(\mathbb{Z}_+,U))$, we have that

$$\|(Hv)(j)\| \le \|Hv\|_{l^2} \le \|H\|\|v\|_{l^2}, \qquad \forall\, j\in\mathbb{Z}_+,\ v\in l^2(\mathbb{Z}_+,U).$$

Combining this with (4.79) gives,

$$\|(H(\varphi\circ u))(j)\| \le \|H\|\|\varphi\circ u\|_{l^2} \le K\|r\|_{m^2}. \tag{4.82}$$

Invoking (4.61) we have,

$$\begin{aligned}\|u(j)\| &\leq \|(H(\varphi\circ u))(j)\| + \|\mathbf{G}(1)\|\|(J(\varphi\circ u))(j)\| + \|r(j)\| \\ &\leq \|(H(\varphi\circ u))(j)\| + \|\mathbf{G}(1)\|\|(J(\varphi\circ u))(j)\| + \|r\|_{m^2}. \qquad (4.83)\end{aligned}$$

Taking the supremum and applying (4.80) combined with (4.82), we obtain from (4.83),

$$\|u\|_{l^\infty} = \sup_{n\geq 0}\|u(n)\| \leq K\|r\|_{m^2}. \qquad (4.84)$$

By (4.60) we have that,

$$\Delta u = \Delta r_1 - G(\varphi\circ u)$$

and since $\varphi\circ u \in l^2(\mathbb{Z}_+,U)$ by (4.79), $G \in \mathscr{B}(l^2(\mathbb{Z}_+,U))$ and $r_1 \in l^2(\mathbb{Z}_+,U)$ we conclude that $\Delta u \in l^2(\mathbb{Z}_+,U)$. Furthermore, it follows from (4.79) that $\|\Delta u\|_{l^2} \leq K\|r\|_{m^2}$. Combining this with, (4.84), (4.81), (4.80), (4.79), it is clear that (4.54) holds.

Proof of Statement 2: By (4.61), equation (4.55) is equivalent to the claim that

$$\lim_{n\to\infty}(H(\varphi\circ u))(n) = 0.$$

Since $H \in \mathscr{B}(l^2(\mathbb{Z}_+,U))$ and $\varphi\circ u \in l^2(\mathbb{Z}_+,U)$, we conclude that $H(\varphi\circ u) \in l^2(\mathbb{Z}_+,U)$. Thus $H(\varphi\circ u)(n) \to 0$ as $n\to\infty$. It is now clear that if $\lim_{n\to\infty}u(n)$ exists, then, by (4.55), $\lim_{n\to\infty}\sum_{j=0}^{n}(\varphi\circ u)(j)$ exists and (4.56) follows trivially.

Proof of Statement 3: From (4.66) and (4.75) we have,

$$\begin{aligned}&\frac{1}{2}\left\|[Q\mathbf{G}(1)]^{1/2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)\right\|^2 \\ &= \frac{1}{2}\|[Q\mathbf{G}(1)]^{1/2}[\mathbf{G}(1)]^{-1}r_2\|^2 - \mathrm{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j), Qu(j)\rangle \\ &-\mathrm{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j), (G_Q(\varphi\circ u))(j)\rangle + \frac{1}{2}\sum_{j=0}^{n}\|[Q\mathbf{G}(1)]^{1/2}(\varphi\circ u)(j)\|^2 \\ &+\mathrm{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j), Qr_1(j)\rangle. \qquad (4.85)\end{aligned}$$

The RHS of (4.85) has a finite limit as $n\to\infty$, since $\mathrm{Re}\,\langle(\varphi\circ u), Qu\rangle, \langle(\varphi\circ u), G_Q(\varphi\circ u)\rangle, \langle(\varphi\circ u), Qr_1\rangle$ and $\|(\varphi\circ u)\|^2$ are in $l^1(\mathbb{Z}_+)$. Hence we have that,

$$\lim_{n\to\infty}\left\|[Q\mathbf{G}(1)]^{1/2}\left(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\right)\right\|$$

exists. Set

$$\lambda := \lim_{n\to\infty} \left\| [Q\mathbf{G}(1)]^{1/2}\Big(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\Big)\right\|. \tag{4.86}$$

Now using (4.55) and writing $f(n) := \sum_{j=0}^{n}(\varphi\circ u)(j)$ we have,

$$\lim_{n\to\infty}(u(n) - r_2 + \mathbf{G}(1)f(n)) = 0. \tag{4.87}$$

Let $g(n) := [Q\mathbf{G}(1)]^{1/2}(f(n)-[\mathbf{G}(1)]^{-1}r_2)$. From (4.86) we have $\lim_{n\to\infty}\|g(n)\| = \lambda$, that is, $g(n)$ approaches the sphere S of radius λ centred at 0 as $n\to\infty$. So by (4.87),

$$0 = \lim_{n\to\infty}(u(n) - r_2 + \mathbf{G}(1)f(n)) = \lim_{n\to\infty}(u(n) + [\mathbf{G}(1)][Q\mathbf{G}(1)]^{-1/2}g(n))$$

and we see that $u(n)$ approaches the set $\mathbf{G}(1)[Q\mathbf{G}(1)]^{-1/2}S$. Finally if $\dim U = 1$, then the sphere S consists of just one or two points. To see that the sequence $u(n)$ does not oscillate between small neighbourhoods of each of these two points, we observe that $(\Delta u)(n)\to 0$ as $n\to\infty$ (since $\Delta u\in l^2(\mathbb{Z}_+, U)$ by statement 1). Therefore we conclude $\lim_{n\to\infty}u(n)$ exists. It is then clear from statement 2 that $\lim_{n\to\infty}\sum_{j=0}^{n}(\varphi\circ u)(j)$ exists.

Proof of Statement 4: In proving statement 4, we use (4.78), the only problem being that the sector condition now only holds whenever $j\geq n_0$. Hence the first term on the LHS of (4.78) could have either sign. However, setting

$$\Gamma := \operatorname{Re}\sum_{j=0}^{n_0}\Big|\langle(\varphi\circ u)(j), Qu(j) - P(\varphi\circ u)(j)\rangle\Big|$$

we see from (4.78) that for all $n\geq n_0$,

$$\begin{aligned}
&\operatorname{Re}\sum_{j=n_0}^{n}\langle(\varphi\circ u)(j), Qu(j) - P(\varphi\circ u)(j)\rangle + \frac{\varepsilon}{2}\sum_{j=0}^{n}\|(\varphi\circ u)(j)\|^2\\
&+\frac{1}{2}\left\|[Q\mathbf{G}(1)]^{1/2}\Big(\sum_{j=0}^{n}(\varphi\circ u)(j) - [\mathbf{G}(1)]^{-1}r_2\Big)\right\|^2\\
&\leq \Gamma + \frac{1}{2\varepsilon}\sum_{j=0}^{n}\|Qr_1(j)\|^2 + \frac{1}{2}\|[Q\mathbf{G}(1)]^{1/2}[\mathbf{G}(1)]^{-1}r_2\|^2.
\end{aligned} \tag{4.88}$$

Now by (4.58), the first term on the LHS of (4.88) is non-negative. Statements 1-3 can be derived from (4.88) by arguments identical to those used previously in this proof with no further changes, except that the RHS of (4.54) is now bounded in terms of $\|r\|_{m^2}$ and Γ.

Proof of Statement 5: Assume that the additional assumptions (B), (C) and (D) are satisfied. Since u is bounded, there exists a closed bounded set $B \subset U$ such that $u(n) \in B$ for all $n \in \mathbb{Z}_+$. It follows from (C) that $\varphi^{-1}(0) \cap B$ is precompact. Let $\eta > 0$. Consequently, for given $\eta > 0$, $\varphi^{-1}(0) \cap B$ is contained in a finite union of open balls with radius η, each ball centred at some point in $\mathrm{cl}(\varphi^{-1}(0) \cap B)$. Denoting this union by B_η, we claim that $u(n) \in B_\eta$ for all sufficiently large n. This is trivially true if $B \subset B_\eta$. If not, then the set $C := B \backslash B_\eta$ is non-empty. Moreover, C is bounded and closed with $\varphi^{-1}(0) \cap C = \emptyset$ and so $\inf_{v \in C} \|\varphi(v)\| > 0$ by (D). We know from statement 1 that $\varphi \circ u \in l^2(\mathbb{Z}_+, U)$, hence $\lim_{n\to\infty}(\varphi \circ u)(n) = 0$, and so also in this case $u(n) \in B_\eta$ for all sufficiently large n. This implies that

$$\lim_{n\to\infty} \mathrm{dist}(u(n), \varphi^{-1}(0) \cap B) = 0 \tag{4.89}$$

and, a fortiori,

$$\lim_{n\to\infty} \mathrm{dist}(u(n), \varphi^{-1}(0)) = 0\,, \tag{4.90}$$

completing the proof of statement 5.

Proof of Statement 6: To verify statement 6, we note that, by (4.89) and precompactness of $\varphi^{-1}(0) \cap B$, the set $\{u(n) : n \in \mathbb{Z}_+\}$ is precompact. Consequently, the ω-limit set Ω of u is non-empty, compact and is approached by $u(n)$ as $n \to \infty$. Invoking (4.90), we see that $\Omega \subset \mathrm{cl}(\varphi^{-1}(0))$. Furthermore, by statement 3, we know that $u(n)$ approaches $\mathbf{G}(1)[Q\mathbf{G}(1)]^{-1/2}S$ for some sphere $S \subset U$ centred at 0, hence $\Omega \subset \mathbf{G}(1)[Q\mathbf{G}(1)]^{-1/2}S$. Hence,

$$\Omega \subset \mathrm{cl}(\varphi^{-1}(0)) \cap \mathbf{G}(1)[Q\mathbf{G}(1)]^{-1/2}S\,. \tag{4.91}$$

Since, by statement 1, $(\Delta u)(n) \to 0$ as $n \to \infty$, it follows from Lemma 2.1.7 that Ω is connected. On the other hand, by hypothesis, $\mathrm{cl}(\varphi^{-1}(0)) \cap \mathbf{G}(1)[Q\mathbf{G}(1)]^{-1/2}S$ is totally disconnected, implying via (4.91) that Ω is totally disconnected. As a consequence, Ω must consist of exactly one point, or, equivalently, $u(n)$ converges as $n \to \infty$. □

We now assume that $\varepsilon = 0$ in the positive-real condition (4.53) and consider the absolute stability problem shown in Figure 4.1.

Theorem 4.2.4. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U)) \cap \mathscr{B}(l^\infty(\mathbb{Z}_+, U))$ be a shift-invariant operator with transfer function $\mathbf{G}$ satisfying assumption* (A) *with $\mathbf{G}(1)$ invertible, and let $\varphi : \mathbb{Z}_+ \times U \to U$ be a time-varying non-linearity. Assume that there exist self-adjoint $P \in \mathscr{B}(U)$, invertible $Q \in \mathscr{B}(U)$ with $Q\mathbf{G}(1) = [Q\mathbf{G}(1)]^* \geq 0$ such* *that* (4.52) *holds and*

$$P+\frac{1}{2}\left[\frac{1}{e^{i\theta}-1}Q\mathbf{G}(e^{i\theta})+\frac{1}{e^{-i\theta}-1}\mathbf{G}^*(e^{i\theta})Q^*\right]\geq 0, \qquad \text{a.a. } \theta\in(0,2\pi). \quad (4.92)$$

Let $r\in m^1(\mathbb{Z}_+,U)$ *and let* $u:\mathbb{Z}_+\to U$ *be the unique solution of* (4.1). *Then the following statements hold.*

1. *There exists a constant* $K>0$ *(which depends only on* P*,* Q *and* G*, but not on* r*) such that,*

$$\|u\|_{l^\infty}+(\|\mathrm{Re}\,\langle(\varphi\circ u),Qu-P(\varphi\circ u)\rangle\|_{l^1})^{1/2} + \sup_{n\geq 0}\left\|\sum_{j=0}^{n}(\varphi\circ u)(j)\right\|\leq K\|r\|_{m^1}. \quad (4.93)$$

2. *Under the additional assumptions* (B), (C) *(see Theorem* 4.2.1*) and*

   (E) $\inf_{v\in B}\mathrm{Re}\,\langle\varphi(v),Qv-P\varphi(v)\rangle>0$ *for every bounded closed set* $B\subset U$ *such that* $\varphi^{-1}(0)\cap B=\emptyset$*,*

   we have that $\lim_{n\to\infty}\mathrm{dist}(u(n),\varphi^{-1}(0))=0$.

3. *If the additional assumptions of statement 2 hold and* φ *is continuous, then* $\lim_{n\to\infty}(\Delta u)(n)=0$*. If, further,* $\varphi^{-1}(0)$ *is totally disconnected, then* $\lim_{n\to\infty}u(n)=:u^\infty$ *exists with* $u^\infty\in\varphi^{-1}(0)$.

Proof. Defining,

$$\sigma(n):=\sum_{j=0}^{n}(\varphi\circ u)(j), \qquad \forall\, n\in\mathbb{Z}_+, \qquad \sigma(-1):=0,$$

and setting $\varepsilon=0$ in (4.72) we obtain,

$$\mathrm{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j),Qu(j)-P(\varphi\circ u)(j)\rangle+\frac{1}{2}\left\|[Q\mathbf{G}(1)]^{1/2}\sigma(n)\right\|^2 \leq \mathrm{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j),Qr(j)\rangle. \quad (4.94)$$

By assumption $r=r_1+r_2\vartheta$, where $r_1\in l^1(\mathbb{Z}_+,U)$ and $r_2\in U$. Using this decomposition of r we can write the right-hand-side of (4.94) as,

$$\operatorname{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j),Qr(j)\rangle=\operatorname{Re}\sum_{j=0}^{n}\langle\sigma(j)-\sigma(j-1),Qr_1(j)\rangle+\operatorname{Re}\langle\sigma(n),Qr_2\rangle. \tag{4.95}$$

Partial summation yields the identity,

$$\begin{aligned}\sum_{j=0}^{n}\langle\sigma(j)-\sigma(j-1),Qr_1(j)\rangle=\sum_{j=0}^{n}&\langle\sigma(j),Qr_1(j)-Qr_1(j+1)\rangle\\&+\langle\sigma(n),Qr_1(n+1)\rangle.\end{aligned} \tag{4.96}$$

Combining (4.96) with (4.95) and (4.94) we obtain,

$$\begin{aligned}&\operatorname{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j),Qu(j)-P(\varphi\circ u)(j)\rangle+\frac{1}{2}\left\|[Q\mathbf{G}(1)]^{1/2}\sigma(n)\right\|^2\\&\leq\operatorname{Re}\sum_{j=0}^{n}\langle\sigma(j),Qr_1(j)-Qr_1(j+1)\rangle+\operatorname{Re}\langle\sigma(n),Qr_1(n+1)\rangle\\&\qquad+\operatorname{Re}\langle\sigma(n),Qr_2\rangle.\end{aligned} \tag{4.97}$$

The RHS of (4.97) can be estimated as follows,

$$\begin{aligned}&\operatorname{Re}\sum_{j=0}^{n}\langle\sigma(j),Qr_1(j)-Qr_1(j+1)\rangle+\operatorname{Re}\langle\sigma(n),Qr_1(n+1)\rangle+\operatorname{Re}\langle\sigma(n),Qr_2\rangle\\&\leq 2\|Qr_1\|_{l^1}\max_{0\leq j\leq n}\|\sigma(j)\|+\|\sigma(n)\|\|Qr_1(n+1)\|+\|\sigma(n)\|\|Qr_2\|\\&\leq\|Q\|\max_{0\leq j\leq n}\|\sigma(j)\|(2\|r_1\|_{l^1}+\|r_1\|_{l^1}+\|r_2\|)\\&\leq 3\|Q\|\max_{0\leq j\leq n}\|\sigma(j)\|\|r\|_{m^1}.\end{aligned} \tag{4.98}$$

Combining (4.98) with (4.97) gives,

$$\begin{aligned}\operatorname{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j),Qu(j)-P(\varphi\circ u)(j)\rangle+\frac{1}{2}\left\|[Q\mathbf{G}(1)]^{1/2}\sigma(n)\right\|^2&\\\leq 3\|Q\|\max_{0\leq j\leq n}\|\sigma(j)\|\|r\|_{m^1}.&\end{aligned} \tag{4.99}$$

Proof of Statement 1: In the following, $K>0$ is a generic constant which will be suitably adjusted in every step and depends only on P, Q and G, but not on n or r.

By sector condition (4.52), the first term on the LHS of (4.99) is non-negative and so, from (4.99) we obtain,

$$\|\sigma(n)\| = \left\| \sum_{j=0}^{n} (\varphi \circ u)(j) \right\| \leq K\|r\|_{m^1}, \qquad \forall\, n \in \mathbb{Z}_+,$$

which in turn implies that,

$$\|\sigma\|_{l^\infty} = \sup_{n\geq 0} \left\| \sum_{j=0}^{n} (\varphi \circ u)(j) \right\| \leq K\|r\|_{m^1}. \tag{4.100}$$

Therefore, using (4.100) in (4.99) shows that,

$$(\|\mathrm{Re}\,\langle(\varphi \circ u), Qu - P(\varphi \circ u)\rangle\|_{l^1})^{1/2} \leq K\|r\|_{m^1}. \tag{4.101}$$

Since G is shift-invariant it commutes with J, and hence from (4.60),

$$\begin{aligned}\|u(n)\| &= \|r(n) - (J(G(\varphi \circ u)))(n)\| \\ &\leq \|r(n)\| + \|(G(J(\varphi \circ u)))(n)\| \\ &\leq \|r\|_{m^1} + \|G\|\|\sigma\|_{l^\infty},\end{aligned}$$

where we have used the fact that $G \in \mathscr{B}(l^\infty(\mathbb{Z}_+, U))$. Taking the supremum and applying (4.100) we obtain,

$$\|u\|_{l^\infty} = \sup_{n\geq 0} \|u(n)\| \leq K\|r\|_{m^1}.$$

Together with (4.101) and (4.100) it is clear that (4.93) holds.

Proof of Statement 2: To prove statement 2, assume that the additional assumptions (B), (C) and (E) are satisfied. Since u is bounded, there exists a closed bounded set $B \subset U$ such that $u(n) \in B$ for all $n \in \mathbb{Z}_+$. It follows from (C) that $\varphi^{-1}(0) \cap B$ is precompact. Let $\eta > 0$. Consequently, for given $\eta > 0$, $\varphi^{-1}(0) \cap B$ is contained in a finite union of open balls with radius η, each ball centred at some point in $\mathrm{cl}(\varphi^{-1}(0) \cap B)$. Denoting this union by B_η, we claim that $u(n) \in B_\eta$ for all sufficiently large n. This is trivially true if $B \subset B_\eta$. If not, then the set $C := B \setminus B_\eta$ is non-empty. Define $\psi : U \to [0, \infty)$ by

$$\psi(v) := \mathrm{Re}\,\langle \varphi(v), Qv - P\varphi(v)\rangle\,.$$

Since C is bounded and closed with $\varphi^{-1}(0) \cap C = \emptyset$, assumption (E) implies that $\inf_{v\in C} \psi(v) > 0$. We know from statement 1 that $\psi \circ u \in l^1(\mathbb{Z}_+)$, hence $\lim_{n\to\infty}(\psi \circ u)(n) = 0$, and so also in this case $u(n) \in B_\eta$ for all sufficiently large

n. This implies that

$$\lim_{n\to\infty} \operatorname{dist}(u(n), \varphi^{-1}(0) \cap B) = 0\,, \tag{4.102}$$

completing the proof of statement 2.

Proof of Statement 3: Finally, to prove statement 3, note that, by precompactness of $\varphi^{-1}(0) \cap B$ and (4.102), the set $\{u(n) : n \in \mathbb{Z}_+\}$ is precompact. This together with (4.102) and the continuity of φ shows that

$$\lim_{n\to\infty} (\varphi \circ u)(n) = 0\,. \tag{4.103}$$

Applying Δ to (4.60) yields

$$\Delta u = \Delta r_1 - (G(\varphi \circ u)). \tag{4.104}$$

To see that

$$\lim_{n\to\infty} (G(\varphi \circ u))(n) = 0, \tag{4.105}$$

we observe the following. Set $v := \varphi \circ u$. Let $\varepsilon > 0$ and let $N_1 \in \mathbb{N}$ be such that,

$$\|v(n)\| \le \frac{\varepsilon}{2\|G\|}\,, \qquad \forall\, n \ge N_1,$$

where we have used (4.103) and the fact that $G \in \mathscr{B}(l^\infty(\mathbb{Z}_+, U))$. Define $v_1 \in F(\mathbb{Z}_+, U)$ by

$$v_1(n) = \begin{cases} 0, & 0 \le n < N_1, \\ v(n), & n \ge N_1, \end{cases}$$

so that $\|v_1\|_{l^\infty} \le \varepsilon/(2\|G\|)$. Defining $v_2 := v - v_1$ we have trivially that $v_2 \in l^2(\mathbb{Z}_+, U)$, so that $Gv_2 \in l^2(\mathbb{Z}_+, U)$. Consequently, there exists $N_2 \in \mathbb{N}$ such that,

$$\|(Gv_2)(n)\| \le \frac{\varepsilon}{2}, \qquad \forall\, n \ge N_2.$$

Finally, since $Gv = Gv_1 + Gv_2$, using the fact that $G \in \mathscr{B}(l^\infty(\mathbb{Z}_+, U))$,

$$\begin{aligned} \|(Gv)(n)\| \le \|(Gv_1)(n)\| + \|(Gv_2)(n)\| &\le \|Gv_1\|_{l^\infty} + \|(Gv_2)(n)\| \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \qquad \forall\, n \ge N_2. \end{aligned}$$

Hence we see that (4.105) holds. Furthermore, $\Delta r_1 \in l^1(\mathbb{Z}_+, U)$, and so, by (4.105) and (4.104),

$$\lim_{n\to\infty} (\Delta u)(n) = 0\,.$$

Consequently, by Lemma 2.1.7, the non-empty and compact ω-limit set Ω of u is connected. Since $\Omega \subset \varphi^{-1}(0)$ and, by hypothesis, $\varphi^{-1}(0)$ is totally disconnected, we obtain that Ω is a singleton, or, equivalently, $u(n)$ converges as $n \to \infty$. $\square$

Taking $P = (1/a)I$ (where $a \in (0, \infty)$) and $Q = I$ a result similar to Corollary 4.2.3 can also be obtained from Theorem 4.2.4.

4.3 The J_0 integrator

We now change the integrator J to an integrator with direct feedthrough. In this subsection we derive results analogous to those in §§4.1 and 4.2 for the J_0 integrator (see, (2.1)). We consider an absolute stability problem for the feedback system shown in Figure 4.2 where φ, G and r are as before.

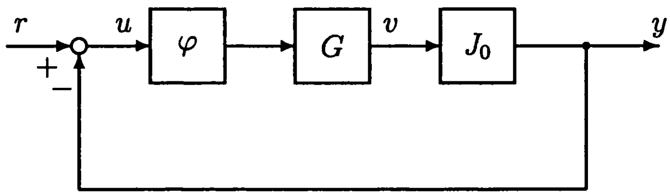

Figure 4.2: Discrete-time feedback system with J_0 integrator

From Figure 4.2 we can derive the following governing equations,

$$u = r - y, \qquad v = G(\varphi \circ u), \qquad y = J_0 v.$$

Equivalently,

$$u = r - (J_0 G)(\varphi \circ u). \tag{4.106}$$

Lemma 3.5.3 provides a condition under which (4.106) has a solution.

Lemma 4.3.1. *For $v \in F(\mathbb{Z}_+, U)$, we have the following formula,*

$$\operatorname{Re} \sum_{n=0}^{m} \left\langle v(n), \sum_{k=0}^{n} v(k) \right\rangle = \frac{1}{2} \left\| \sum_{k=0}^{m} v(k) \right\|^2 + \frac{1}{2} \sum_{k=0}^{m} \|v(k)\|^2, \qquad \forall\, m \in \mathbb{Z}_+.$$

Proof. The case $m = 0$ is clear. For $m \in \mathbb{N}$, noting that

$$\operatorname{Re} \sum_{n=0}^{m} \left\langle v(n), \sum_{k=0}^{n} v(k) \right\rangle = \operatorname{Re} \sum_{n=1}^{m} \left\langle v(n), \sum_{k=0}^{n-1} v(k) \right\rangle + \sum_{k=0}^{m} \|v(k)\|^2,$$

the claim follows from Lemma 4.1.1. $\square$

We begin by deriving results analogous to those in §4.1. We assume that $U = \mathbb{R}$, that φ is time-independent and consider the absolute stability problem shown in Figure 4.2. The following result is a stability criterion of Popov-type in an input-output context.

Theorem 4.3.2. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ be a shift-invariant operator with transfer function $\mathbf{G}$ satisfying assumption* (A) *and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a measurable non-decreasing non-linearity. Assume that $\mathbf{G}(1) > 0$ and there exists numbers $q \geq 0$, $\varepsilon > 0$ and $a \in (0, \infty]$ such that* (4.2) *holds and*

$$\frac{1}{a} + \operatorname{Re}\left[\left(q + \frac{e^{i\theta}}{e^{i\theta} - 1}\right)\mathbf{G}(e^{i\theta})\right] \geq \varepsilon, \qquad \text{a.a. } \theta \in (0, 2\pi). \tag{4.107}$$

Let $r \in m^2(\mathbb{Z}_+, \mathbb{R})$ and let $u : \mathbb{Z}_+ \to \mathbb{R}$ be a solution of (4.106). *Then the conclusions of Theorem* 4.1.2 *hold.*

Proof. We have,

$$u + (J_0 G)(\varphi \circ u) = r, \tag{4.108}$$

or equivalently,

$$\triangle_0 u + G(\varphi \circ u) = \triangle_0 r. \tag{4.109}$$

We now write (4.108) in a slightly more convenient form, namely,

$$u + H(\varphi \circ u) + \mathbf{G}(1) J_0(\varphi \circ u) = r, \tag{4.110}$$

where $H := J_0 \circ G - \mathbf{G}(1) J_0$. It is clear that this operator is shift-invariant with transfer function $\mathbf{H}$ given by,

$$\mathbf{H}(z) = \frac{z}{z-1}[\mathbf{G}(z) - \mathbf{G}(1)], \qquad z \in \mathbb{E}_1. \tag{4.111}$$

From assumption (A) and from the fact that $\mathbf{G} \in H^\infty(\mathbb{E}_1)$, we conclude that $\mathbf{H} \in H^\infty(\mathbb{E}_1)$, and hence $H \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$.

We multiply (4.109) by q and to this add (4.110) to obtain

$$\begin{aligned} q(\triangle_0 u)(j) + u(j) \;+\; & G_q(\varphi \circ u)(j) + \mathbf{G}(1)(J_0(\varphi \circ u))(j) \\ &= q(\triangle_0 r)(j) + r(j), \qquad \forall\, j \in \mathbb{Z}_+, \end{aligned} \tag{4.112}$$

where we have defined the operator G_q by

$$G_q := qG + H.$$

Invoking (4.111), we see that the transfer function $\mathbf{G}_q$ of G_q is given by

$$\mathbf{G}_q(z) := q\mathbf{G}(z) + \frac{z}{z-1}[\mathbf{G}(z) - \mathbf{G}(1)], \qquad z \in \mathbb{E}_1.$$

Multiplying through by $(\varphi \circ u)(j)$ and summing from 0 to n in (4.112) yields,

$$\begin{aligned} q\sum_{j=0}^{n}&(\varphi \circ u)(j)(\Delta_0 u)(j) + \sum_{j=0}^{n}(\varphi \circ u)(j)u(j) \\ &+ \sum_{j=0}^{n}(\varphi \circ u)(j)(G_q(\varphi \circ u))(j) + \mathbf{G}(1)\sum_{j=0}^{n}(\varphi \circ u)(j)\sum_{k=0}^{j}(\varphi \circ u)(k) \\ &\qquad = q\sum_{j=0}^{n}(\varphi \circ u)(j)(\Delta_0 r)(j) + \sum_{j=0}^{n}(\varphi \circ u)(j)r(j). \end{aligned} \tag{4.113}$$

An application of Lemma 4.3.1 to the last term on the LHS of (4.113) yields,

$$\begin{aligned} q\sum_{j=0}^{n}&(\varphi \circ u)(j)(\Delta_0 u)(j) + \sum_{j=0}^{n}(\varphi \circ u)(j)u(j) + \sum_{j=0}^{n}(\varphi \circ u)(j)(G_q(\varphi \circ u))(j) \\ &+ \frac{\mathbf{G}(1)}{2}\left(\sum_{j=0}^{n}(\varphi \circ u)(j)\right)^2 + \frac{\mathbf{G}(1)}{2}\sum_{j=0}^{n}(\varphi \circ u)^2(j) \\ &= q\sum_{j=0}^{n}(\varphi \circ u)(j)(\Delta_0 r)(j) + \sum_{j=0}^{n}(\varphi \circ u)(j)r(j). \end{aligned} \tag{4.114}$$

We note that,

$$\begin{aligned} \operatorname{Re}\mathbf{G}_q(z) &= \operatorname{Re}\left(q\mathbf{G}(z) + \frac{z}{z-1}[\mathbf{G}(z) - \mathbf{G}(1)]\right) \\ &= \operatorname{Re}\left[\left(q + \frac{z}{z-1}\right)\mathbf{G}(z)\right] - \mathbf{G}(1)\operatorname{Re}\frac{z}{z-1}. \end{aligned}$$

Taking $z = e^{i\theta}$ with $\theta \in (0, 2\pi)$,

$$\begin{aligned}\operatorname{Re}\frac{z}{z-1} = \operatorname{Re}\left(\frac{z(\overline{z-1})}{|z-1|^2}\right) = \operatorname{Re}\left(\frac{e^{i\theta}(\overline{e^{i\theta}-1})}{|e^{i\theta}-1|^2}\right) &= \frac{\cos\theta(\cos\theta - 1) + \sin^2\theta}{(\cos\theta-1)^2 + \sin^2\theta} \\ &= \frac{\cos^2\theta + \sin^2\theta - \cos\theta}{2(1-\cos\theta)} \\ &= \frac{1-\cos\theta}{2(1-\cos\theta)} \\ &= \frac{1}{2},\end{aligned}$$

and we obtain,

$$\operatorname{Re}\mathbf{G}_q(e^{i\theta}) = \operatorname{Re}\left[\left(q + \frac{e^{i\theta}}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right] - \frac{\mathbf{G}(1)}{2}, \qquad \text{a.a. } \theta \in (0, 2\pi). \quad (4.115)$$

Combining (4.115) with (4.107) we see that,

$$\begin{aligned}\operatorname{Re}\mathbf{G}_q(e^{i\theta}) &= \operatorname{Re}\left[\left(q + \frac{e^{i\theta}}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right] - \frac{\mathbf{G}(1)}{2} \\ &\geq \varepsilon - \frac{1}{a} - \frac{\mathbf{G}(1)}{2}.\end{aligned} \quad (4.116)$$

Using (4.116) and arguments similar to those used to derive (4.18) in the proof of Theorem 4.1.2, it follows that,

$$\sum_{j=0}^{n}(\varphi\circ u)(j)(G_q(\varphi\circ u))(j) \geq \left(\varepsilon - \frac{1}{a} - \frac{\mathbf{G}(1)}{2}\right)\sum_{j=0}^{n}(\varphi\circ u)^2(j). \quad (4.117)$$

Applying (4.117) to (4.114) we obtain (4.19). Arguments identical to those used in the proof of Theorem 4.1.2 can now be invoked to complete the proof. □

We now assume that $\varepsilon = 0$ in the positive-real condition (4.107) and consider the absolute stability problem shown in Figure 4.2.

Theorem 4.3.3. *Let $G \in \mathscr{B}(l^\infty(\mathbb{Z}_+, \mathbb{R}))$ be a shift-invariant operator with transfer function $\mathbf{G}$ satisfying assumption* (A) *and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a measurable non-decreasing non-linearity. Assume that $\mathbf{G}(1) > 0$ and there exists numbers $q \geq 0$ and $a \in (0, \infty]$ such that* (4.2) *holds and*

$$\frac{1}{a} + \operatorname{Re}\left[\left(q + \frac{e^{i\theta}}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right] \geq 0, \qquad \text{a.a. } \theta \in (0, 2\pi). \quad (4.118)$$

Let $r \in m^1(\mathbb{Z}_+, \mathbb{R})$, and let $u : \mathbb{Z}_+ \to \mathbb{R}$ be a solution of (4.106). *Then the* *conclusion of Theorem* 4.1.3 *holds.*

Proof. Obtaining (4.19) as in the proof of Theorem 4.3.2 and setting $\varepsilon = 0$ in (4.19), the claim can be proved by invoking the same arguments used in the proof of Theorem 4.1.3 with J replaced by J_0. □

We now derive results analogous to those in §4.2. We consider the absolute stability problem in Figure 4.2 where φ is time-dependent.

Theorem 4.3.4. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ be a shift-invariant operator with transfer function $\mathbf{G}$ satisfying assumption* (A) *with $\mathbf{G}(1)$ invertible, and let $\varphi : \mathbb{Z}_+ \times U \to U$ be a time-varying non-linearity. Assume that there exist self-adjoint $P \in \mathscr{B}(U)$, invertible $Q \in \mathscr{B}(U)$ with $Q\mathbf{G}(1) = [Q\mathbf{G}(1)]^* \geq 0$ and a number $\varepsilon > 0$ such that* (4.52) *holds and*

$$P + \frac{1}{2}\left[\frac{e^{i\theta}}{e^{i\theta}-1}Q\mathbf{G}(e^{i\theta}) + \frac{e^{-i\theta}}{e^{-i\theta}-1}\mathbf{G}^*(e^{i\theta})Q^*\right] \geq \varepsilon I, \quad \text{a.a. } \theta \in (0, 2\pi). \tag{4.119}$$

Let $r \in m^2(\mathbb{Z}_+, U)$, and let $u : \mathbb{Z}_+ \to U$ be a solution of (4.106). *Then statements 1-6 of Theorem* 4.2.1 *hold.*

Before proving Theorem 4.3.4, we state a slightly simplified version of this result (where P and Q are scalars and $P \geq 0$) in the form of a corollary which is convenient in the context of applications of Theorem 4.3.4 to integral control (see Chapter 5).

Corollary 4.3.5. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ be a shift-invariant operator with transfer function $\mathbf{G}$ satisfying assumption* (A) *with $\mathbf{G}(1)$ invertible, $\mathbf{G}(1) = \mathbf{G}^*(1) \geq 0$, and let $\varphi : \mathbb{Z}_+ \times U \to U$ be a time-varying non-linearity satisfying* (4.59) *for some $a \in (0, \infty]$. Assume that there exists a number $\varepsilon > 0$ such that,*

$$\frac{1}{a}I + \frac{1}{2}\left[\frac{e^{i\theta}}{e^{i\theta}-1}\mathbf{G}(e^{i\theta}) + \frac{e^{-i\theta}}{e^{-i\theta}-1}\mathbf{G}^*(e^{i\theta})\right] \geq \varepsilon I, \qquad \text{a.a. } \theta \in (0, 2\pi). \tag{4.120}$$

Then for all $r \in m^2(\mathbb{Z}_+, U)$, the conclusions of Theorem 4.3.4 *hold with $P = (1/a)I$ and $Q = I$.*

Proof of Theorem 4.3.4. We have,

$$u + (J_0 G)(\varphi \circ u) = r. \tag{4.121}$$

We now write (4.121) in a slightly more convenient form, namely,

$$u + H(\varphi \circ u) + \mathbf{G}(1)J_0(\varphi \circ u) = r, \tag{4.122}$$

where $H := J_0 \circ G - \mathbf{G}(1) J_0$. It is clear that this operator is shift-invariant with transfer function $\mathbf{H}$ given by,

$$\mathbf{H}(z) = \frac{z}{z-1}[\mathbf{G}(z) - \mathbf{G}(1)], \qquad z \in \mathbb{E}_1. \tag{4.123}$$

From assumption (A) and from the fact that $\mathbf{G} \in H^\infty(\mathbb{E}_1, \mathscr{B}(U))$, we conclude that $\mathbf{H} \in H^\infty(\mathbb{E}_1, \mathscr{B}(U))$, and hence $H \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$.

Applying Q to (4.122) we obtain,

$$Qu(j) + (G_Q(\varphi \circ u))(j) + Q\mathbf{G}(1)(J_0(\varphi \circ u))(j) = Qr(j), \qquad \forall\, j \in \mathbb{Z}_+, \tag{4.124}$$

where we have defined the operator G_Q by $G_Q := QH$. Invoking (4.123) we see that the transfer function $\mathbf{G}_Q$ of G_Q is given by

$$\mathbf{G}_Q := \frac{z}{z-1} Q[\mathbf{G}(z) - \mathbf{G}(1)], \qquad z \in \mathbb{E}_1.$$

Forming the inner product with $(\varphi \circ u)(j)$, taking real parts and summing from 0 to n in (4.124) yields,

$$\begin{aligned}
\mathrm{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qu(j) \rangle &+ \mathrm{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), (G_Q(\varphi \circ u))(j) \rangle \\
&+ \mathrm{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Q\mathbf{G}(1) \sum_{k=0}^{j} (\varphi \circ u)(k) \rangle \\
&= \mathrm{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Qr(j) \rangle.
\end{aligned} \tag{4.125}$$

Since, by assumption, $Q\mathbf{G}(1) = [Q\mathbf{G}(1)]^* \geq 0$, the square root $[Q\mathbf{G}(1)]^{1/2}$ of $Q\mathbf{G}(1)$ exists and hence, an application of Lemma 4.3.1 to the last term on the LHS of (4.125) yields,

$$\begin{aligned}
&\mathrm{Re} \sum_{j=0}^{n} \langle (\varphi \circ u)(j), Q\mathbf{G}(1) \sum_{k=0}^{j} (\varphi \circ u)(k) \rangle \\
&= \mathrm{Re} \sum_{j=0}^{n} \langle [Q\mathbf{G}(1)]^{1/2} (\varphi \circ u)(j), \sum_{k=0}^{j} [Q\mathbf{G}(1)]^{1/2} (\varphi \circ u)(k) \rangle \\
&= \frac{1}{2} \left\| [Q\mathbf{G}(1)]^{1/2} \sum_{j=0}^{n} (\varphi \circ u)(j) \right\|^2 + \frac{1}{2} \sum_{j=0}^{n} \| [Q\mathbf{G}(1)]^{1/2} (\varphi \circ u)(j) \|^2.
\end{aligned} \tag{4.126}$$

Combining (4.126) with (4.125) gives,

$$\begin{aligned}
\operatorname{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j), Qu(j)\rangle + \operatorname{Re}&\sum_{j=0}^{n}\langle(\varphi\circ u)(j), (G_Q(\varphi\circ u))(j)\rangle \\
&+\frac{1}{2}\left\|[Q\mathbf{G}(1)]^{1/2}\sum_{j=0}^{n}(\varphi\circ u)(j)\right\|^2 \\
&+\frac{1}{2}\sum_{j=0}^{n}\|[Q\mathbf{G}(1)]^{1/2}(\varphi\circ u)(j)\|^2 \\
&= \operatorname{Re}\sum_{j=0}^{n}\langle(\varphi\circ u)(j), Qr(j)\rangle. \qquad (4.127)
\end{aligned}$$

We note that,

$$\begin{aligned}
\mathbf{G}_Q(z)+\mathbf{G}_Q^*(z) &= \frac{z}{z-1}Q[\mathbf{G}(z)-\mathbf{G}(1)] + \left(\frac{z}{z-1}Q[\mathbf{G}(z)-\mathbf{G}(1)]\right)^* \\
&= \frac{z}{z-1}Q\mathbf{G}(z) + \frac{\overline{z}}{\overline{z}-1}\mathbf{G}^*(z)Q^* - \left(\frac{z}{z-1}+\frac{\overline{z}}{\overline{z}-1}\right)Q\mathbf{G}(1).
\end{aligned}$$

Taking $z=e^{i\theta}$ with $\theta\in(0,2\pi)$ yields,

$$\frac{e^{i\theta}}{e^{i\theta}-1}+\frac{e^{-i\theta}}{e^{-i\theta}-1} = \frac{e^{i\theta}(e^{-i\theta}-1)+e^{-i\theta}(e^{i\theta}-1)}{(e^{i\theta}-1)(e^{-i\theta}-1)} = \frac{2-e^{-i\theta}-e^{i\theta}}{2-e^{i\theta}-e^{-i\theta}} = 1$$

and we obtain,

$$\begin{aligned}
\mathbf{G}_Q(e^{i\theta})+\mathbf{G}_Q^*(e^{i\theta}) = \frac{e^{i\theta}}{e^{i\theta}-1}&Q\mathbf{G}(e^{i\theta}) + \frac{e^{-i\theta}}{e^{-i\theta}-1}\mathbf{G}^*(e^{i\theta})Q^* \\
&- Q\mathbf{G}(1), \quad \text{a.a. } \theta\in(0,2\pi). \qquad (4.128)
\end{aligned}$$

Combining (4.128) with (4.119) we see that,

$$\frac{1}{2}[\mathbf{G}_Q(e^{i\theta})+\mathbf{G}_Q^*(e^{i\theta})] \geq \varepsilon I - P - \frac{Q\mathbf{G}(1)}{2}. \qquad (4.129)$$

Using (4.129) and arguments similar to those used to derive (4.71) in the proof of Theorem 4.2.1, it follows that,

$$\begin{aligned}
\operatorname{Re} \sum_{j=0}^{n} &\langle (\varphi \circ u)(j), (G_Q(\varphi \circ u))(j) \rangle \\
&\geq \varepsilon \sum_{j=0}^{n} \|(\varphi \circ u)(j)\|^2 - \sum_{j=0}^{n} \langle (\varphi \circ u)(j), P(\varphi \circ u)(j) \rangle \\
&\qquad - \frac{1}{2} \sum_{j=0}^{n} \|[Q\mathbf{G}(1)]^{1/2}(\varphi \circ u)(j)\|^2. \qquad (4.130)
\end{aligned}$$

Applying (4.130) to (4.127), we obtain (4.72). Arguments identical to those used in the proof of Theorem 4.2.1 can now be invoked to complete the proof. □

We now assume that $\varepsilon = 0$ in the positive-real condition (4.119) and consider the absolute stability problem shown in Figure 4.2.

Theorem 4.3.6. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U)) \cap \mathscr{B}(l^\infty(\mathbb{Z}_+, U))$ be a shift-invariant operator with transfer function $\mathbf{G}$ satisfying assumption* (A) *with $\mathbf{G}(1)$ invertible, and let $\varphi : \mathbb{Z}_+ \times U \to U$ be a time-varying non-linearity. Assume that there exist self-adjoint $P \in \mathscr{B}(U)$, invertible $Q \in \mathscr{B}(U)$ with $Q\mathbf{G}(1) = [Q\mathbf{G}(1)]^* \geq 0$ such that* (4.52) *holds and*

$$P + \frac{1}{2}\left[\frac{e^{i\theta}}{e^{i\theta} - 1} Q\mathbf{G}(e^{i\theta}) + \frac{e^{-i\theta}}{e^{-i\theta} - 1}\mathbf{G}^*(e^{i\theta})Q^*\right] \geq 0, \qquad \text{a.a. } \theta \in (0, 2\pi). \quad (4.131)$$

Let $r \in m^1(\mathbb{Z}_+, U)$, and let $u : \mathbb{Z}_+ \to U$ be a solution of (4.106). *Then the conclusions of Theorem* 4.2.4 *hold.*

Proof. Obtaining (4.72) as in the proof of Theorem 4.3.4 and setting $\varepsilon = 0$ in (4.72), the claim can be proved by invoking the same arguments used in the proof of Theorem 4.2.4 with J replaced by J_0. □

As before taking $P = (1/a)I$ (where $a \in (0, \infty]$) and $Q = I$ a result similar to Corollary 4.3.5 can also be obtained from Theorem 4.3.6.

4.4 Incremental sector conditions

For ease of application (see Chapter 10), the results in this section are stated for the J_0 integrator.

In this section we derive versions of Theorems 4.3.4 and 4.3.6 which yield stability properties of the difference of two solutions of (4.106). In this context, the following incremental sector condition

$$\begin{aligned}&\operatorname{Re}\langle\varphi(n,\xi_1)-\varphi(n,\xi_2),Q(\xi_1-\xi_2)\rangle\\&\geq\langle\varphi(n,\xi_1)-\varphi(n,\xi_2),P(\varphi(n,\xi_1)-\varphi(n,\xi_2))\rangle,\ \ \forall\, n\in\mathbb{Z}_+,\ \xi_1,\xi_2\in U\end{aligned}\tag{4.132}$$

is relevant. Note that if φ is unbiased, that is,

$$\varphi(n,0)=0,\qquad n\in\mathbb{Z}_+,$$

then trivially (4.52) is implied by (4.132).

Corollary 4.4.1. *Let $G\in\mathscr{B}(l^2(\mathbb{Z}_+,U))$ be a shift-invariant operator with transfer function $\mathbf{G}$ satisfying assumption* (A) *with $\mathbf{G}(1)$ invertible and let $\varphi:\mathbb{Z}_+\times U\to U$ be a time-varying non-linearity. Assume that there exist self-adjoint $P\in\mathscr{B}(U)$, invertible $Q\in\mathscr{B}(U)$ with $Q\mathbf{G}(1)=[Q\mathbf{G}(1)]^*\geq 0$ and a number $\varepsilon>0$ such that* (4.132) *and* (4.53) *hold. Let $r_1,r_2\in F(\mathbb{Z}_+,U)$ and suppose that $r_1-r_2\in m^2(\mathbb{Z}_+,U)$, that is, $r_1-r_2=v_1+v_2\vartheta$ with $v_1\in l^2(\mathbb{Z}_+,U)$ and $v_2\in U$. Let $u_1:\mathbb{Z}_+\to U$, $u_2:\mathbb{Z}_+\to U$ be solutions of* (4.106) *corresponding to forcing functions r_1 and r_2, respectively. Then the following statements hold.*

1. *There exists a constant K (which depends only on ε, P, Q and G, but not on r_1 or r_2) such that,*

$$\begin{aligned}&\|u_1-u_2\|_{l^\infty}+\|\Delta(u_1-u_2)\|_{l^2}+\|\varphi\circ u_1-\varphi\circ u_2\|_{l^2}\\&+(\|\operatorname{Re}\langle\varphi\circ u_1-\varphi\circ u_2,Q(u_1-u_2)\rangle\|_{l^1})^{1/2}\\&+\sup_{n\geq 0}\left\|\sum_{j=0}^{n}(\varphi\circ u_1)(j)-(\varphi\circ u_2)(j)\right\|\leq K\|r_1-r_2\|_{m^2}.\end{aligned}\tag{4.133}$$

2. *We have,*

$$\lim_{n\to\infty}\left(u_1(n)-u_2(n)+\mathbf{G}(1)\sum_{j=0}^{n}((\varphi\circ u_1)(j)-(\varphi\circ u_2)(j))\right)=v_2;$$

in particular, $\lim_{n\to\infty}(u_1(n)-u_2(n))$ exists if and only if

$$\lim_{n\to\infty}\sum_{j=0}^{n}((\varphi\circ u_1)(j)-(\varphi\circ u_2)(j))$$

exists, in which case

$$\lim_{n\to\infty}(u_1(n)-u_2(n))=v_2-\mathbf{G}(1)\lim_{n\to\infty}\sum_{j=0}^{n}((\varphi\circ u_1)(j)-(\varphi\circ u_2)(j)).$$

3. There exists a sphere $S \subset U$ centred at 0, such that

$$\lim_{n\to\infty} \operatorname{dist}(u_1(n) - u_2(n), \mathbf{G}(1)[Q\mathbf{G}(1)]^{-1/2}S) = 0,$$

in particular, if $\dim U = 1$, then $\lim_{n\to\infty}(u_1(n) - u_2(n))$ exists.

4. If we relax condition (4.132) and only require that for some $n_0 > 0$,

$$\begin{aligned}&\operatorname{Re}\langle \varphi(n,\xi_1) - \varphi(n,\xi_2), Q(\xi_1 - \xi_2)\rangle \\ &\geq \langle \varphi(n,\xi_1) - \varphi(n,\xi_2), P(\varphi(n,\xi_1) - \varphi(n,\xi_2))\rangle, \ \ \forall\, n \geq n_0, \ \ \xi_1, \xi_2 \in U,\end{aligned}$$

then the LHS of (4.133) is still finite (but no longer bounded in terms of $\|r_1 - r_2\|_{m^2}$) and statements 2 and 3 remain valid.

Proof. By (4.106)

$$u_1 - u_2 = r_1 - r_2 - (J_0G)(\varphi \circ u_1 - \varphi \circ u_2). \tag{4.134}$$

Define $\psi : \mathbb{Z}_+ \times U \to U$ by

$$\psi(n,\xi) := \varphi(n, \xi + u_2(n)) - \varphi(n, u_2(n)), \qquad (n,\xi) \in \mathbb{Z}_+ \times U. \tag{4.135}$$

Then ψ is unbiased,

$$\psi(n, u_1(n) - u_2(n)) = \varphi(n, u_1(n)) - \varphi(n, u_2(n)), \quad \forall\, n \in \mathbb{Z}_+,$$

and it follows from (4.132) that,

$$\begin{aligned}&\operatorname{Re}\langle \psi(n,\xi), Q\xi\rangle \\ &= \operatorname{Re}\langle \varphi(n,\xi + u_2(n)) - \varphi(n,u_2(n)), Q\xi\rangle \\ &\geq \langle \varphi(n,\xi + u_2(n)) - \varphi(n,u_2(n)), P(\varphi(n,\xi + u_2(n)) - \varphi(n,u_2(n)))\rangle \\ &= \langle \psi(n,\xi), P\psi(n,\xi)\rangle, \qquad (n,\xi) \in \mathbb{Z}_+ \times U. \end{aligned} \tag{4.136}$$

It now follows from (4.134), with $r := r_1 - r_2$ that,

$$u_1 - u_2 = r - (J_0G)(\psi \circ (u_1 - u_2)). \tag{4.137}$$

Since by assumption $r = r_1 - r_2 \in m^2(\mathbb{Z}_+, U)$ and, by (4.136), ψ satisfies (4.52) and, all the other assumptions of Theorem 4.3.4 are satisfied, we may apply Theorem 4.3.4 to equation (4.137). Statements 1-4 now follow from statements 1-4 of Theorem 4.3.4. □

Corollary 4.4.2. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U)) \cap \mathscr{B}(l^\infty(\mathbb{Z}_+, U))$ be a shift-invariant operator with transfer function $\mathbf{G}$ satisfying assumption* (A) *with $\mathbf{G}(1)$ invertible* *and let $\varphi : \mathbb{Z}_+ \times U \to U$ be a time-varying non-linearity. Assume that there exist self-adjoint $P \in \mathscr{B}(U)$, invertible $Q \in \mathscr{B}(U)$ with $Q\mathbf{G}(1) = [Q\mathbf{G}(1)]^* \geq 0$ such that* (4.132) *and* (4.131) *hold. Let $r_1, r_2 \in F(\mathbb{Z}_+, U)$ and suppose that $r_1 - r_2 \in m^1(\mathbb{Z}_+, U)$. Let $u_1 : \mathbb{Z}_+ \to U$, $u_2 : \mathbb{Z}_+ \to U$ be solutions of* (4.106) *corresponding to forcing functions r_1 and r_2, respectively. Then, there exists a constant K (which depends only on P, Q and G, but not on r_1 or r_2) such that,*

$$\|u_1 - u_2\|_{l^\infty} + (\|\mathrm{Re}\,\langle \varphi \circ u_1 - \varphi \circ u_2, Q(u_1 - u_2) - P(\varphi \circ u_1 - \varphi \circ u_2)\rangle\|_{l^1})^{1/2}$$
$$+ \sup_{n \geq 0} \left\| \sum_{j=0}^{n} (\varphi \circ u_1)(j) - (\varphi \circ u_2)(j) \right\| \leq K \|r_1 - r_2\|_{m^1}. \qquad (4.138)$$

Proof. By assumption $r = r_1 - r_2 \in m^1(\mathbb{Z}_+, U)$. Consequently, ψ (as defined by (4.135)) satisfies (4.52) and, all the other assumptions of Theorem 4.3.6 are satisfied. Applying Theorem 4.3.6 to (4.137), we obtain (4.138). □

Remarks 4.4.3. (a) It is also possible to obtain versions of Theorems 4.2.1 and 4.2.4 with incremental sector conditions.

(b) Suppose that φ is time-independent. Then ψ given by (4.135) is time-varying. Consequently, using the approach in this section, it seems difficult to obtain versions of Theorems 4.1.2, 4.1.3, 4.3.2 and 4.3.3 with incremental sector conditions. ◇

4.5 Notes and references

While most of the available absolute-stability literature is devoted to continuous-time systems, there are still a considerable number of references which treat discrete-time systems; see, for example, [15], [23], [28], [58] and [62].

In the finite-dimensional case there are a number of results available in the literature, many of which have been obtained by Lyapunov techniques applied to state-space models with the positive real lemma playing a crucial role, see, for example, [23], [28] and [58]. In the infinite-dimensional case, often an input-output approach is taken. There are several distinguishing features of the stability results in this chapter. Firstly, the linear system is the series interconnection of an input-output stable linear system and an integrator (meaning, in particular, that the linear system is not input-output stable), such systems are often referred to as 'marginally stable'. Furthermore, the non-linearities in this chapter satisfy a sector condition with lower-gain possibly zero, which, in particular, allows for saturation and deadzone effects. By contrast, in the discrete-time input-output absolute stability results contained in [15] and [62], if the lower-gain is allowed to be zero, the linear part is assumed to be input-output stable (in particular, it is not allowed to contain an integrator); see, for example, [15], pp. 99-108, 191-194, and [62], pp. 370-371 . Furthermore, we remark that in the discrete-time input-output absolute stability results available in the literature the assumptions imposed on the impulse response of the linear system (where it is usually assumed that the convolution kernel of G is in l^1) are more restrictive than in most of the results in this chapter the only exceptions being Theorems 4.1.3 and 4.3.3. We note that in particular, small-gain arguments can not be used to prove the results in §§4.1-4.4.

The results in §§4.1-4.3 form the discrete-time analogues of the continuous-time absolute stability results contained in [9] and [10]. Note that in contrast to the continuous-time Popov-type results of [9] and [10], which were obtained in an infinite-dimensional Hilbert space setting, the results in §4.1 are only stated for the special case $U = \mathbb{R}$. The reason for this restriction is we need to obtain a positive lower bound for the term $\sum_{j=0}^{n}(\varphi \circ u)(j)(\Delta_0 u)(j)$, (see proof of Theorem 4.1.2). To obtain this bound we assume that φ is measurable, so that φ is integrable, and non-decreasing and make use of the intermediate-value theorem. Consequently, it seems difficult to extend such an approach to a general Hilbert space setting. As a relatively minor point, the circle-criterion-type results in §4.2 are obtained in an infinite-dimensional complex Hilbert space setting, the analogous continuous-time results of [9] and [10] only being obtained for real Hilbert spaces. We observe that with $q = 0$ in the circle-criterion-type results we can consider time-varying non-linearities whereas, in the Popov-type results only time-independent non-linearities can be considered.

We now make some further comments on the results in §§4.1-4.3. The stability results in §§4.1-4.3 which impose a strict positive real condition (that is, $\varepsilon > 0$) allow us to choose reference values r from the space $m^2(\mathbb{Z}_+, U)$. In contrast to this, the stability results in §§4.1-4.3 which impose a non-strict positive real condition (that is, $\varepsilon = 0$) only allow us to choose reference values r from $m^1(\mathbb{Z}_+, U)$, a smaller space than $m^2(\mathbb{Z}_+, U)$.

In contrast to the results in §§4.1 and 4.2, the results in §4.3 (that is, results containing an integrator with direct feedthrough) hold for a larger class of sector bounded non-linearities. In particular, the Popov-type results in §4.3 hold for sector bounded non-linearities with upper sector bound $a = \infty$, allowing us to consider cubic non-linearities, a situation which was not possible in §4.1.

The results in §4.4 provide us with versions of Theorems 4.3.4 and 4.3.6 with incremental sector conditions, yielding stability properties of the difference of two solutions of (4.106). In particular, if we consider the linear system in Figure 4.2 as an operator $r \mapsto u$, the results in §4.4 give us some kind of global continuity property of the system. The corresponding results in §4.3 only give us a continuity property at 0.

The results in §§4.1-4.3 with strict positive real condition form the basis of §2 of [5].

Chapter 5

Low-gain integral control of infinite-dimensional discrete-time systems in the presence of input/output non-linearities

In this chapter we apply Theorem 4.1.2, Theorem 4.3.2, Corollary 4.2.3 and Corollary 4.3.5 to derive results on low-gain integral control with output disturbances in the single-input-single-output setting.

5.1 Integral control in the presence of input non-linearities

Consider the feedback system shown in Figure 5.1, where $\rho \in \mathbb{R}$ is a constant reference value, $k \in \mathbb{R}$ is a gain parameter, $u^0 \in \mathbb{R}$ is the initial state of the integrator (or, equivalently, the initial value of u), $\varphi : \mathbb{R} \to \mathbb{R}$ is a static input non-linearity and $G \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ is a shift-invariant operator with transfer function denoted by $\mathbf{G}$. The function g models the effect of non-zero initial conditions of the system with input-output operator G and the function d is an external disturbance.

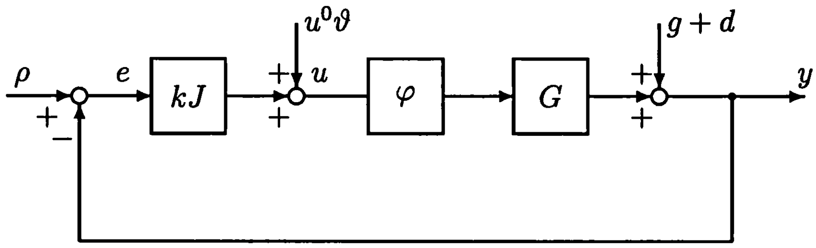

Figure 5.1: Low-gain control problem with constant gain and J integrator

From Figure 5.1 we can derive the following governing equations,

$$y = g + d + G(\varphi \circ u), \qquad e = \rho\vartheta - y, \qquad u = u^0\vartheta + kJe,$$

or, equivalently

$$u = u^0\vartheta + kJ(\rho\vartheta - g - d - G(\varphi \circ u)). \tag{5.1}$$

Since u^0, k, ρ, g and d are given, it is clear from Proposition 3.5.1 that (5.1) has a unique solution. Trivially, (5.1) can be written as an initial-value problem,

$$(\Delta u)(n) = k[\rho - (g(n) + d(n) + (G(\varphi \circ u))(n))], \;\; n \geq 1, \;\; u(0) = u^0 \in \mathbb{R}.$$

The aim in this section is to choose the gain parameter k such that the tracking error,

$$e(n) := \rho - y(n) = \rho - (g(n) + d(n) + (G(\varphi \circ u))(n)) = (\Delta u)(n)/k$$

converges to 0 as $n \to \infty$. We define,

$$f_J(G) := \sup_{q \geq 0} \left\{ \text{ess inf}_{\theta \in (0,2\pi)} \text{Re} \left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta} - 1} \right) \mathbf{G}(e^{i\theta}) \right] \right\}. \tag{5.2}$$

If the transfer function $\mathbf{G}$ of G satisfies assumption (A), then $-\infty < f_J(G) \leq -\mathbf{G}(1)/2$ (see Appendix 2, Proposition 12.1.3 (i), for more details).

For $a \in (0, \infty)$ we let $\mathscr{S}(a)$ denote the set of all functions $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying the sector condition

$$0 \leq \varphi(v)v \leq av^2, \qquad \forall\, v \in \mathbb{R}.$$

We denote by $\mathscr{S}(\infty)$ the set of all functions $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying,

$$0 \leq \varphi(v)v, \qquad \forall\, v \in \mathbb{R}.$$

Lemma 5.1.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous and non-decreasing. Let $\xi \in \mathbb{R}$ and define $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$ by*

$$\widetilde{\varphi}(w) := \varphi(w + \xi) - \varphi(\xi), \qquad \forall\, w \in \mathbb{R}.$$

Then $\widetilde{\varphi} \in \mathscr{S}(\infty)$ and, if $\varphi - \varphi(0) \in \mathscr{S}(a)$ for some $a \in (0, \infty)$, then there exists $b \in (0, \infty)$ such that $\widetilde{\varphi} \in \mathscr{S}(b)$.

Proof. It follows easily from the fact that φ is non-decreasing that

$$\widetilde{\varphi}(w)w \geq 0, \qquad \forall\, w \in \mathbb{R},$$

showing in particular that $\widetilde{\varphi} \in \mathscr{S}(\infty)$. Assume now that $\varphi - \varphi(0) \in \mathscr{S}(a)$ for some $a \in (0, \infty)$. Write

$$\begin{aligned} \widetilde{\varphi}(w)w &= \varphi(w + \xi)w - \varphi(\xi)w \\ &= (\varphi(w + \xi) - \varphi(0))(w + \xi) + \varphi(0)(w + \xi) \\ &\qquad - \varphi(w + \xi)\xi - \varphi(\xi)w, \quad w \in \mathbb{R}. \end{aligned}$$

It follows that

$$\widetilde{\varphi}(w)w \leq a(w + \xi)^2 + \varphi(0)(w + \xi) - \varphi(w + \xi)\xi - \varphi(\xi)w, \qquad w \in \mathbb{R}.$$

Consequently,

$$\widetilde{\varphi}(w)w \leq \left(a + \frac{2a\xi}{w} + \frac{a\xi^2}{w^2} + \frac{\varphi(0)}{w} + \frac{\varphi(0)\xi}{w^2} - \frac{\varphi(w + \xi)\xi}{w^2} - \frac{\varphi(\xi)}{w} \right) w^2, \quad w \in \mathbb{R}.$$

Noting that φ is linearly bounded, it follows that for all $\delta > 0$ there exists $R > 0$ such that

$$\widetilde{\varphi}(w)w \leq (a + \delta)w^2, \qquad |w| \geq R. \tag{5.3}$$

Now suppose that $|w| < R$. Since φ is locally Lipschitz continuous and non-decreasing it follows that there exists $\lambda > 0$ such that,

$$\begin{aligned} 0 \leq \widetilde{\varphi}(w)w = |\widetilde{\varphi}(w)w| &\leq |\varphi(w + \xi) - \varphi(\xi)||w| \\ &\leq \lambda|w|^2 = \lambda w^2, \quad |w| < R. \end{aligned} \tag{5.4}$$

Combining (5.3) and (5.4) we see that there exists $b \in (0, \infty)$ such that,

$$\widetilde{\varphi}(w)w \leq bw^2, \quad \forall\, w \in \mathbb{R},$$

showing that $\widetilde{\varphi} \in \mathscr{S}(b)$ for some $b \in (0, \infty)$. □

Theorem 5.1.2. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ be a shift-invariant operator with transfer function $\mathbf{G}$. Assume that assumption (A′) holds with $\mathbf{G}(1) > 0$, that $Jg \in$*

$m^2(\mathbb{Z}_+,\mathbb{R})$, $d = d_1 + d_2\vartheta$ with $Jd_1 \in m^2(\mathbb{Z}_+,\mathbb{R})$ and $d_2 \in \mathbb{R}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous and non-decreasing. Let $\rho \in \mathbb{R}$, assume that $(\rho - d_2)/\mathbf{G}(1) \in \mathrm{im}\varphi$ and let u be the solution of (5.1). *Under these conditions the following statements hold.*

1. *Assume that $\varphi - \varphi(0) \in \mathscr{S}(a)$ for some $a \in (0,\infty)$. Then there exists a constant $k^* \in (0,\infty)$ (depending on G, φ and ρ) such that for all $k \in (0,k^*)$, the limit $\lim_{n\to\infty} u(n) =: u^\infty$ exists and satisfies $\varphi(u^\infty) = (\rho - d_2)/\mathbf{G}(1)$,*
$$e = \Delta u/k \in l^2(\mathbb{Z}_+,\mathbb{R}) \quad \textit{and} \quad \varphi \circ u - \varphi(u^\infty)\vartheta \in l^2(\mathbb{Z}_+,\mathbb{R}).$$
In particular, $\lim_{n\to\infty} e(n) = 0$.

2. *Assume that φ is globally Lipschitz continuous with Lipschitz constant $\lambda > 0$. Then the conclusions of statement* 1 *are valid with $k^* = 1/|\lambda f_J(G)|$.*

Remarks 5.1.3. (i) Theorem 5.1.2 ensures that the tracking error is in $l^2(\mathbb{Z}_+,\mathbb{R})$ and consequently, $e(n)$ converges to 0 as $n \to \infty$.

(ii) Note that in statement 2 of Theorem 5.1.2 the constant k^* depends only on G and the Lipschitz constant of φ, but not on ρ.

(iii) Let $f \in F(\mathbb{Z}_+,\mathbb{R})$. Then $Jf \in m^2(\mathbb{Z}_+,\mathbb{R})$ if and only if $(Jf)(n)$ converges to a finite limit as $n \to \infty$ and $n \mapsto \sum_{k=n}^{\infty} f(k)$ is in $l^2(\mathbb{Z}_+,\mathbb{R})$ (see Appendix 3 Proposition 12.1.5 for more details). Hence sufficient conditions for g and d_1 to satisfy the assumptions in Theorem 5.1.2 are that $(Jg)(n), (Jd_1)(n)$ converge to finite limits as $n \to \infty$ and the functions $n \mapsto \sum_{k=n}^{\infty} g(k)$, $n \mapsto \sum_{k=n}^{\infty} d_1(k)$ are in $l^2(\mathbb{Z}_+,\mathbb{R})$.

(iv) If $f \in F(\mathbb{Z}_+,\mathbb{R})$ is such that $j \mapsto f(j)j^\alpha$ is in $l^2(\mathbb{Z}_+,\mathbb{R})$ for some $\alpha > 1$ then $f \in l^1(\mathbb{Z}_+,\mathbb{R})$ and $n \mapsto \sum_{j=n}^{\infty} |f(j)| \in l^2(\mathbb{Z}_+,\mathbb{R})$ (see Appendix 3, Proposition 12.1.6 for more details). Hence sufficient conditions for g and d_1 to satisfy the assumptions in Theorem 5.1.2 are that, $j \mapsto g(j)j^\beta$ is in $l^2(\mathbb{Z}_+,\mathbb{R})$ and $j \mapsto d_1(j)j^\gamma$ is in $l^2(\mathbb{Z}_+,\mathbb{R})$ for some $\beta, \gamma > 1$.

(v) In general d_2 is unknown, but it is reasonable to assume that $d_2 \in [a,b]$, where a and b are known constants. The condition
$$(\rho - a)/\mathbf{G}(1), (\rho - b)/\mathbf{G}(1) \in \mathrm{im}\varphi$$
does not involve d_2 and is sufficient for $(\rho - d_2)/\mathbf{G}(1)$ to be in $\mathrm{im}\varphi$. Furthermore, if φ is continuous, $\lim_{\xi\to\infty} \varphi(\xi) = \infty$ and $\lim_{\xi\to-\infty} \varphi(\xi) = -\infty$, the assumption $(\rho - d_2)/\mathbf{G}(1) \in \mathrm{im}\varphi$ is automatically satisfied. ◇

Theorem 5.1.2 gives an input-output point of view of the low-gain integral control problem with input non-linearities and output disturbances. We emphasize that Theorem 5.1.2 applies to strongly stable state-space systems (see Chapter 6).

Proof of Theorem 5.1.2. Chose some $u^\rho \in \mathbb{R}$ such that $\varphi(u^\rho) = (\rho - d_2)/\mathbf{G}(1)$ (such a u^ρ exists, since, by assumption, $(\rho - d_2)/\mathbf{G}(1) \in \mathrm{im}\varphi$). For any $k \in \mathbb{R}$ we know that (5.1) has a unique solution u defined on $\mathbb{Z}_+$. We define a function $v : \mathbb{Z}_+ \to \mathbb{R}$ by,

$$v := u - u^\rho \vartheta. \tag{5.5}$$

Moreover, we define $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$ by,

$$\widetilde{\varphi}(\xi) := \varphi(\xi + u^\rho) - \varphi(u^\rho), \qquad \forall\, \xi \in \mathbb{R}. \tag{5.6}$$

Then by (5.1),

$$\begin{aligned}
v &= u^0\vartheta - u^\rho\vartheta + kJ(\rho\vartheta - g - d) - kJ(G(\varphi(u - u^\rho + u^\rho))\\
&= u^0\vartheta - u^\rho\vartheta + kJ(\rho\vartheta - g - d) - kJ(G(\varphi(v + u^\rho)))\\
&= u^0\vartheta - u^\rho\vartheta + kJ(\rho\vartheta - g - d) - kJ\varphi(u^\rho)G\vartheta - kJ(G(\widetilde{\varphi} \circ v)).
\end{aligned}$$

Hence it follows that,

$$v = r - kJ(G(\widetilde{\varphi} \circ v)), \tag{5.7}$$

where the function r is given by

$$r = u^0\vartheta - u^\rho\vartheta - kJ(g + d_1 + \varphi(u^\rho)G\vartheta - (\rho - d_2)\vartheta). \tag{5.8}$$

In order to apply Theorem 4.1.2 to equation (5.7), we first show that $r \in m^2(\mathbb{Z}_+, \mathbb{R})$, that is, r satisfies the relevant assumption in Theorem 4.1.2. Since by assumption $Jg, Jd_1 \in m^2(\mathbb{Z}_+, \mathbb{R})$, we immediately see that

$$u^0\vartheta - u^\rho\vartheta - kJ(g + d_1) \in m^2(\mathbb{Z}_+, \mathbb{R}).$$

Therefore by (5.8) it is sufficient to show that the function

$$n \mapsto (J(\varphi(u^\rho)G\vartheta - (\rho - d_2)\vartheta))(n) = \varphi(u^\rho)(J(G\vartheta - \mathbf{G}(1)\vartheta))(n)$$

belongs to $m^2(\mathbb{Z}_+, \mathbb{R})$, where we have used that $\varphi(u^\rho) = (\rho - d_2)/\mathbf{G}(1)$. Note that

$$J(G\vartheta - \mathbf{G}(1)\vartheta) = J(G\vartheta - \mathbf{G}(1)\vartheta) - \mathbf{G}'(1)\vartheta + \mathbf{G}'(1)\vartheta. \tag{5.9}$$

By assumption (A$'$), $J(G\vartheta - \mathbf{G}(1)\vartheta) - \mathbf{G}'(1)\vartheta \in l^2(\mathbb{Z}_+, \mathbb{R})$. Hence it follows from (5.9) that $J(G\vartheta - \mathbf{G}(1)\vartheta) \in m^2(\mathbb{Z}_+, \mathbb{R})$.

Proof of Statement 1: Since φ is non-decreasing and $\varphi - \varphi(0) \in \mathscr{S}(a)$ for some $a \in (0, \infty)$, it follows from Lemma 5.1.1 that there exists $b \in (0, \infty)$ such that $\widetilde{\varphi} \in \mathscr{S}(b)$. Define $k^* := 1/|bf_J(G)|$, let $k \in (0, k^*)$ and set $\varepsilon := \frac{1}{2}(1/b + kf_J(G))$.

Then $\varepsilon > 0$ and we can choose some $q > 0$ such that

$$\operatorname{ess\,inf}_{\theta\in(0,2\pi)}\mathrm{Re}\left[\left(\frac{q}{e^{i\theta}}+\frac{1}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right] \geq f_J(G)-\frac{\varepsilon}{k}=\frac{(\varepsilon-\frac{1}{b})}{k}. \tag{5.10}$$

Thus

$$\frac{1}{b}+\mathrm{Re}\left[\left(\frac{q}{e^{i\theta}}+\frac{1}{e^{i\theta}-1}\right)k\mathbf{G}(e^{i\theta})\right]\geq\varepsilon, \qquad \text{a.a. } \theta\in(0,2\pi),$$

and (4.3) holds with a replaced by b and $\mathbf{G}$ replaced by $k\mathbf{G}$. An application of Theorem 4.1.2 to (5.7) now yields that $\lim_{n\to\infty} v(n)$ exists and is finite, $\widetilde{\varphi}\circ v\in l^2(\mathbb{Z}_+,\mathbb{R})$ and $\Delta_0 v\in l^2(\mathbb{Z}_+,\mathbb{R})$. Consequently, we have $\lim_{n\to\infty} u(n) =: u^\infty$ exists and is finite,

$$\varphi(u^\infty)=\varphi(u^\rho)=(\rho-d_2)/\mathbf{G}(1),$$

and

$$\varphi\circ u-\varphi(u^\infty)\in l^2(\mathbb{Z}_+,\mathbb{R}).$$

We note that $\Delta v=\Delta u$ and since $\Delta_0 v\in l^2(\mathbb{Z}_+,\mathbb{R})$ we have $\Delta u\in l^2(\mathbb{Z}_+,\mathbb{R})$. It now follows that $e=\Delta u/k\in l^2(\mathbb{Z}_+,\mathbb{R})$ and hence, $\lim_{n\to\infty} e(n)=0$.

Proof of Statement 2: By hypothesis φ is globally Lipschitz. Denoting the Lipschitz constant of φ by $\lambda > 0$, it follows that

$$\widetilde{\varphi}(v)v=[\varphi(v+u^\rho)-\varphi(u^\rho)]v\leq\lambda|(v+u^\rho)-u^\rho||v|=\lambda v^2, \quad \forall\, v\in\mathbb{R}.$$

Since φ is non-decreasing it is also clear that $\widetilde{\varphi}(v)v\geq 0$. Consequently, it follows that $\widetilde{\varphi}\in\mathscr{S}(\lambda)$. Now the arguments in the proof of statement 1 apply with b replaced by λ. □

We now change the integrator J to J_0 and consider the feedback system shown in Figure 5.2, where the function g models the effect of non-zero initial conditions of the system with input-output operator G and $c\in\mathbb{R}$ is the initial state of the integrator.

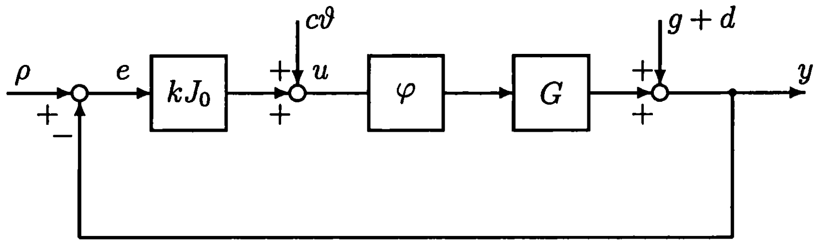

Figure 5.2: Low-gain control problem with constant gain and J_0 integrator

From Figure 5.2 we can derive the following governing equations,

$$y = g + d + G(\varphi \circ u), \qquad e = \rho\vartheta - y, \qquad u = c\vartheta + kJ_0 e,$$

or, equivalently

$$u = c\vartheta + kJ_0(\rho\vartheta - (g + d + G(\varphi \circ u)). \tag{5.11}$$

For given c, k, ρ, g and d, it follows from Proposition 3.5.3 that (5.11) has at least one solution (a unique solution, respectively) if the map $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(\xi) := \xi + k\mathbf{G}(\infty)\varphi(\xi), \qquad \xi \in \mathbb{R},$$

is surjective (bijective, respectively). Since $\triangle_0 J_0 = J_0 \triangle_0 = I$, (5.11) is equivalent to

$$\triangle_0 u = c\delta + k[\rho\vartheta - (g + d + G(\varphi \circ u))].$$

We define,

$$f_{J_0}(G) := \sup_{q \geq 0} \left\{ \operatorname{ess\,inf}_{\theta \in (0,2\pi)} \mathrm{Re} \left[\left(q + \frac{e^{i\theta}}{e^{i\theta} - 1} \right) \mathbf{G}(e^{i\theta}) \right] \right\}. \tag{5.12}$$

If the transfer function $\mathbf{G}$ of G satisfies assumption (A), then $-\infty < f_{J_0}(G) \leq \infty$ (see Appendix 2, Proposition 12.1.4 (ii), for more details).

Theorem 5.1.4. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ be a shift-invariant operator with transfer function $\mathbf{G}$. Assume that assumption (A$'$) holds with $\mathbf{G}(1) > 0$, that $J_0 g \in m^2(\mathbb{Z}_+, \mathbb{R})$, $d = d_1 + d_2\vartheta$ with $J_0 d_1 \in m^2(\mathbb{Z}_+, \mathbb{R})$ and $d_2 \in \mathbb{R}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous and non-decreasing. Let $\rho \in \mathbb{R}$, assume that $(\rho - d_2)/\mathbf{G}(1) \in \operatorname{im}\varphi$ and let u be a solution of (5.11). Under these conditions the following statements hold.*

1. *Assume that $\varphi - \varphi(0) \in \mathscr{S}(a)$ for some $a \in (0, \infty)$. Then there exists a constant $k^* \in (0, \infty]$ (depending on G, φ and ρ) such that for all $k \in (0, k^*)$, the limit $\lim_{n \to \infty} u(n) =: u^\infty$ exists and satisfies $\varphi(u^\infty) = (\rho - d_2)/\mathbf{G}(1)$,*

   $$e = \frac{1}{k}(\triangle_0 u - c\delta) \in l^2(\mathbb{Z}_+, \mathbb{R}) \quad \text{and} \quad \varphi \circ u - \varphi(u^\infty)\vartheta \in l^2(\mathbb{Z}_+, \mathbb{R}).$$

   In particular, $\lim_{n \to \infty} e(n) = 0$. If $f_{J_0}(G) = 0$, then the above conclusions are valid with $k^* = \infty$.

2. *Assume that φ is globally Lipschitz continuous with Lipschitz constant $\lambda > 0$. Then the conclusions of statement 1 are valid with $k^* = 1/|\lambda f_{J_0}(G)|$, where $1/0 := \infty$.*

3. *Under the assumption that $f_{J_0}(G) > 0$, the conclusions of statement 1 are valid with $k^* = \infty$.*

Remark 5.1.5. Note that the range of gains guaranteed to achieve tracking (specified by k^*) is potentially larger than in Theorem 5.1.2. In particular, $k^* = \infty$ is feasible if $f_{J_0}(G) \geq 0$. $\diamond$

Proof of Theorem 5.1.4. Chose some $u^\rho \in \mathbb{R}$ such that $\varphi(u^\rho) = (\rho - d_2)/\mathbf{G}(1)$ (such a u^ρ exists, since, by assumption, $(\rho - d_2)/\mathbf{G}(1) \in \mathrm{im}\varphi$). Let u be a solution of (5.11). Let $v : \mathbb{Z}_+ \to \mathbb{R}$ and $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$ be as defined in (5.5) and (5.6). A straightforward calculation invoking (5.11) and similar to the argument leading to (5.7) in the proof of Theorem 5.1.2, shows that v satisfies

$$v = r - kJ_0(G(\widetilde{\varphi} \circ v)) \tag{5.13}$$

where the function r is given by,

$$r = c\vartheta - u^\rho \vartheta - k(J_0(g + d_1 + \varphi(u^\rho)G\vartheta - (\rho - d_2)\vartheta). \tag{5.14}$$

In order to apply Theorem 4.3.2 to equation (5.13) we first show that $r \in m^2(\mathbb{Z}_+, \mathbb{R})$ that is, r satisfies the relevant assumption in Theorem 4.3.2. This can been shown in a similar way to the corresponding argument in the proof of Theorem 5.1.2.

Proof of Statement 1: Since φ is non-decreasing and $\varphi - \varphi(0) \in \mathscr{S}(a)$ for some $a \in (0, \infty)$, it follows from Lemma 5.1.1 that there exists $b \in (0, \infty)$ such that $\widetilde{\varphi} \in \mathscr{S}(b)$. Define

$$k^* := \begin{cases} 1/|bf_{J_0}(G)|, & \text{if } f_{J_0}(G) < 0, \\ \infty, & \text{if } f_{J_0}(G) \geq 0. \end{cases}$$

We may assume that $f_{J_0}(G) < \infty$ because the case $f_{J_0}(G) = \infty$ is included in statement 3. Therefore $k^* \in (0, \infty]$. Let $k \in (0, k^*)$ and set $\varepsilon := \frac{1}{2}(1/b + kf_{J_0}(G))$. Then $\varepsilon > 0$ and we can choose some $q > 0$ such that

$$\operatorname{ess\,inf}_{\theta \in (0,2\pi)} \mathrm{Re}\left[\left(q + \frac{e^{i\theta}}{e^{i\theta} - 1}\right)\mathbf{G}(e^{i\theta})\right] \geq f_{J_0}(G) - \frac{\varepsilon}{k} = \frac{(\varepsilon - \frac{1}{b})}{k}. \tag{5.15}$$

Thus

$$\frac{1}{b} + \mathrm{Re}\left[\left(q + \frac{e^{i\theta}}{e^{i\theta} - 1}\right)k\mathbf{G}(e^{i\theta})\right] \geq \varepsilon, \qquad \text{a.a. } \theta \in (0, 2\pi),$$

and (4.107) holds with a replaced by b and $\mathbf{G}$ replaced by $k\mathbf{G}$. An application of Theorem 4.3.2 to (5.13) now yields that $\lim_{n\to\infty} v(n)$ exists and is finite, $\widetilde{\varphi} \circ v \in l^2(\mathbb{Z}_+, \mathbb{R})$ and $\Delta_0 v \in l^2(\mathbb{Z}_+, \mathbb{R})$. Consequently, we have $\lim_{n\to\infty} u(n) =: u^\infty$ exists and is finite,

$$\varphi(u^\infty) = \varphi(u^\rho) = (\rho - d_2)/\mathbf{G}(1),$$

and

$$\varphi \circ u - \varphi(u^\infty) \in l^2(\mathbb{Z}_+, \mathbb{R}).$$

We note that $\Delta v = \Delta u$ and since $\Delta_0 v \in l^2(\mathbb{Z}_+, \mathbb{R})$ we have $\Delta v \in l^2(\mathbb{Z}_+, \mathbb{R})$ and hence $\Delta u \in l^2(\mathbb{Z}_+, \mathbb{R})$. It is then clear that $\Delta_0 u \in l^2(\mathbb{Z}_+, \mathbb{R})$ and so

$$e = \frac{1}{k}(\Delta_0 u - c\delta) \in l^2(\mathbb{Z}_+, \mathbb{R}),$$

and hence, $\lim_{n\to\infty} e(n) = 0$. It remains to show that if $f_{J_0}(G) = 0$, then the above conclusions are valid with $k^* = \infty$. If $f_{J_0}(G) = 0$, then for every $k > 0$ there exists $q > 0$ such that (5.15) holds.

Proof of Statement 2: As in the proof of statement 2 of Theorem 5.1.2, we can show that $\varphi \in \mathscr{S}(\lambda)$. Now the arguments in the proof of statement 1 apply with b replaced by λ.

Proof of Statement 3: It is clear by the non-decreasing property of φ that $\widetilde{\varphi}(v)v \geq 0$ for all $v \in \mathbb{R}$ and so $\widetilde{\varphi} \in \mathscr{S}(\infty)$. Since $f_{J_0}(G) > 0$, we can choose some $q \geq 0$ such that

$$\operatorname{ess\,inf}_{\theta\in(0,2\pi)} \mathrm{Re}\left[\left(q + \frac{e^{i\theta}}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right] \geq \frac{1}{2} f_{J_0}(G).$$

This implies that, for any $k > 0$,

$$\mathrm{Re}\left[\left(q + \frac{e^{i\theta}}{e^{i\theta}-1}\right)k\mathbf{G}(e^{i\theta})\right] \geq \frac{1}{2} f_{J_0}(G), \qquad \text{a.a. } \theta \in (0, 2\pi).$$

Therefore (4.107) holds with $a = \infty$, $\mathbf{G}$ replaced by $k\mathbf{G}$ and ε replaced by $\frac{1}{2}kf_{J_0}(G)$. As in the proof of statement 1, the claim now follows from Theorem 4.3.2. □

5.2 Integral control in the presence of input and output non-linearities

In this subsection we generalise the feedback scheme in § 5.1 to allow for a time-varying gain and non-linearities in the output as well as in the input.

Consider the feedback system shown in Figure 5.3, where $\kappa : \mathbb{Z}_+ \to \mathbb{R}$ is a time-varying gain, the operator $G \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ is shift-invariant with transfer function denoted by $\mathbf{G}$, $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are static input and output non-linearities, respectively, $\rho \in \mathbb{R}$ is a constant reference value, $u^0 \in \mathbb{R}$ is the initial state of the integrator (or, equivalently, the initial value of u), the function g models the effect of non-zero initial conditions of the system with input-output operator G and the function d is an external disturbance.

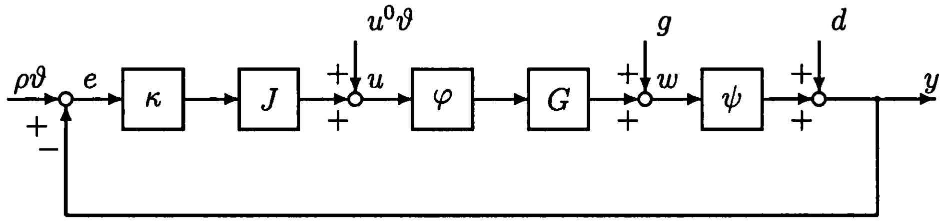

Figure 5.3: Low-gain control problem with time-varying gain and J integrator

From Figure 5.3 we can derive the following governing equations,

$$w = g + G(\varphi \circ u), \qquad y = \psi(w) + d, \qquad u = u^0\vartheta + J(\kappa(\rho\vartheta - y)),$$

or, equivalently

$$u = u^0\vartheta + J(\kappa(\rho\vartheta - d - \psi(g + G(\varphi \circ u)))). \tag{5.16}$$

Since u^0, κ, ρ, g, d and ψ are given, it is clear from Proposition 3.5.1 that (5.16) has a unique solution. Trivially, (5.16) can be written as an initial-value problem,

$$\left.\begin{aligned}(\Delta u)(n) &= \kappa(n)(\rho - d(n) - \psi(g(n) + (G(\varphi \circ u))(n))), \qquad n \geq 1,\\ u(0) &= u^0 \in \mathbb{R}.\end{aligned}\right\} \tag{5.17}$$

The objective is to determine gain functions κ such that the tracking error

$$e(n) := \rho - y(n) = \rho - d(n) - \psi(g(n) + (G(\varphi \circ u))(n))$$

converges to 0 as $n \to \infty$. We introduce the set of feasible reference values

$$\mathscr{R}(G, \varphi, \psi) := \{\psi(\mathbf{G}(1)v) \mid v \in \overline{\operatorname{im}\varphi}\}.$$

It is clear that $\mathscr{R}(G, \varphi, \psi)$ is an interval provided that φ and ψ are continuous.

The following result shows that if φ is continuous and ψ is continuous and monotone, then $\rho \in \mathscr{R}(G, \varphi, \psi)$ is close to being a necessary condition for tracking insofar as, if tracking of ρ is achievable, whilst maintaining boundedness of $\varphi \circ u$ and w, then $\rho \in \mathscr{R}(G, \varphi, \psi)$.

Anticipating the treatment in Chapter 6 we obtain the following proposition.

Proposition 5.2.1. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ be shift-invariant with transfer function $\mathbf{G}$ satisfying assumption* (A). *Assume further, that $\mathbf{G}$ has a power stable state-space realisation. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous and $\psi : \mathbb{R} \to \mathbb{R}$ be continuous and monotone. Let $u \in F(\mathbb{Z}_+, \mathbb{R})$ be such that $\varphi \circ u \in l^\infty(\mathbb{Z}_+, \mathbb{R})$ and*

$g \in F(\mathbb{Z}_+, \mathbb{R})$ be such that $g(n) \to 0$ as $n \to \infty$. If $w \in F(\mathbb{Z}_+, \mathbb{R})$ given by $w = g + G(\varphi \circ u)$ is such that

$$\lim_{n\to\infty} \psi(w(n)) = \rho, \tag{5.18}$$

then $\rho \in \mathscr{R}(G, \varphi, \psi)$.

The proof of Proposition 5.2.1 can be found in Appendix 4.

We define,

$$f_{0,J}(G) := \operatorname{ess\,inf}_{\theta\in(0,2\pi)} \mathrm{Re}\left[\frac{\mathbf{G}(e^{i\theta})}{e^{i\theta}-1}\right]. \tag{5.19}$$

If the transfer function $\mathbf{G}$ of G satisfies assumption (A), then $-\infty < f_{0,J}(G) \leq -\mathbf{G}(1)/2$ (see Appendix 2, Proposition 12.1.3 (ii), for more details).

Theorem 5.2.2. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ be a shift-invariant operator with transfer function $\mathbf{G}$. Assume that assumption (A) holds with $\mathbf{G}(1) > 0$, $g \in l^2(\mathbb{Z}_+, \mathbb{R})$, $d = d_1 + d_2\vartheta$ with $d_1 \in l^1(\mathbb{Z}_+, \mathbb{R})$, $d_2 \in \mathbb{R}$ and $n \mapsto \sum_{j=n}^{\infty} |d_1(j)| \in l^2(\mathbb{Z}_+, \mathbb{R})$, $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are non-decreasing and globally Lipschitz continuous with Lipschitz constants $\lambda_1 > 0$ and $\lambda_2 > 0$, $\rho - d_2 \in \mathscr{R}(G, \varphi, \psi)$ and $\kappa : \mathbb{Z}_+ \to \mathbb{R}$ is bounded and non-negative with*

$$\limsup_{n\to\infty} \kappa(n) < 1/|\lambda_1\lambda_2 f_{0,J}(G)|. \tag{5.20}$$

Let $u : \mathbb{Z}_+ \to \mathbb{R}$ be the unique solution of (5.16). Then the following statements hold.

1. *The limit $(\varphi \circ u)^\infty := \lim_{n\to\infty} \varphi(u(n))$ exists and is finite and*

$$\Delta(\varphi \circ u) \in l^2(\mathbb{Z}_+, \mathbb{R}).$$

2. *The signals $w = g + G(\varphi \circ u)$ and $y = \psi \circ w + d$ have finite limits satisfying*

$$\lim_{n\to\infty} w(n) = \mathbf{G}(1)(\varphi \circ u)^\infty, \qquad \lim_{n\to\infty} y(n) = \psi(\mathbf{G}(1)(\varphi \circ u)^\infty) + d_2.$$

3. *If $\kappa \notin l^1(\mathbb{Z}_+, \mathbb{R})$, then $\lim_{n\to\infty} y(n) = \rho$, or equivalently, $\lim_{n\to\infty} e(n) = 0$.*

4. *If $\rho - d_2$ is an interior point of $\mathscr{R}(G, \varphi, \psi)$, then u is bounded.*

Remarks 5.2.3. (i) A sufficient condition for d_1 to satisfy the assumptions in Theorem 5.2.2 is given in Remark 5.1.3 (iv).

(ii) Note that it is not necessary to know $f_{0,J}(G)$ or the constant λ in order to apply Theorem 5.2.2. If κ is chosen such that $\kappa(n) \to 0$ as $n \to \infty$ and

$\kappa \notin l^1(\mathbb{Z}_+, \mathbb{R})$ (e.g. $\kappa(n) = (1+n)^{-p}$ with $p \in (0,1]$), then the conclusions of statement 3 hold. However, from a practical point of view, gain functions κ with $\lim_{n\to\infty} \kappa(n) = 0$ might not be appropriate, since the system essentially operates in open loop as $n \to \infty$.

(iii) In general d_2 is unknown, but it is reasonable to assume that $d_2 \in [a, b]$, where a and b are known constants. The condition

$$\rho - a, \rho - b \in \mathscr{R}(G, \varphi, \psi)$$

does not involve d_2 and is sufficient for $\rho - d_2$ to be in $\mathscr{R}(G, \varphi, \psi)$. Furthermore, if φ and ψ are continuous, $\lim_{\xi\to\infty} \varphi(\xi) = \infty$, $\lim_{\xi\to-\infty} \varphi(\xi) = -\infty$, $\lim_{\xi\to\infty} \psi(\xi) = \infty$ and $\lim_{\xi\to-\infty} \psi(\xi) = -\infty$, $\mathscr{R}(G, \varphi, \psi)$ is the whole real line and it is clear that $\rho - d_2 \in \mathscr{R}(G, \varphi, \psi)$. $\diamond$

Theorem 5.2.2 gives an input-output point of view of the low-gain integral control problem with input and output non-linearities as well as output disturbances. We emphasize that Theorem 5.2.2 applies to strongly stable state-space systems (see Chapter 6).

Proof of Theorem 5.2.2. Let $u : \mathbb{Z}_+ \to \mathbb{R}$ be the unique solution of (5.16). We shall prove Theorem 5.2.2 by applying Corollary 4.2.3 to the equation satisfied by the input signal

$$w = g + G(\varphi \circ u)$$

of the output non-linearity ψ, modified with an offset which depends on ρ and d_2. Since $\rho - d_2 \in \mathscr{R}(G, \varphi, \psi)$, there exists $\varphi^\rho \in \overline{\mathrm{im}\varphi}$ satisfying

$$\psi(\mathbf{G}(1)\varphi^\rho) = \rho - d_2.$$

We define

$$\begin{aligned} \widetilde{w} &:= w - \mathbf{G}(1)\varphi^\rho\vartheta = g + G(\varphi \circ u) - \mathbf{G}(1)\varphi^\rho\vartheta, \\ \widetilde{\psi}(\xi) &:= \psi(\xi + \mathbf{G}(1)\varphi^\rho) - \rho + d_2, \qquad \forall\, \xi \in \mathbb{R}. \end{aligned} \tag{5.21}$$

Note that $\widetilde{\psi}(0) = 0$. We observe that,

$$\begin{aligned} \widetilde{\psi}(\xi)\xi &= [\psi(\xi + \mathbf{G}(1)\varphi^\rho) - \rho + d_2]\xi \\ &= [\psi(\xi + \mathbf{G}(1)\varphi^\rho) - \psi(\mathbf{G}(1)\varphi^\rho)]\xi, \qquad \forall\, \xi \in \mathbb{R}. \end{aligned}$$

By the non-decreasing property of ψ we see that $[\psi(\xi+\mathbf{G}(1)\varphi^\rho) - \psi(\mathbf{G}(1)\varphi^\rho)]\xi \geq 0$ and hence $\widetilde{\psi}(\xi)\xi \geq 0$ for all $\xi \in \mathbb{R}$. Since ψ is globally Lipschitz with Lipschitz constant λ_2,

$$\begin{aligned}\widetilde{\psi}(\xi)\xi &= [\psi(\xi + \mathbf{G}(1)\varphi^\rho) - \psi(\mathbf{G}(1)\varphi^\rho)]\xi \\ &\le \lambda_2|\xi + \mathbf{G}(1)\varphi^\rho - \mathbf{G}(1)\varphi^\rho||\xi| \\ &= \lambda_2\xi^2, \qquad \forall\, \xi \in \mathbb{R}.\end{aligned}$$

Hence we obtain,

$$0 \le \widetilde{\psi}(\xi)\xi \le \lambda_2\xi^2, \qquad \forall\, \xi \in \mathbb{R}. \tag{5.22}$$

Using (5.17),

$$\Delta u = \kappa(\rho\vartheta - d - \psi(g + G(\varphi \circ u))) = \kappa(\rho\vartheta - d - \psi(\widetilde{w} + \mathbf{G}(1)\varphi^\rho)) = -\kappa(d_1 + \widetilde{\psi} \circ \widetilde{w}).$$

Defining

$$b_u(n) := \begin{cases} \frac{(\Delta(\varphi\circ u))(n)}{(\Delta u)(n)}, & \text{if } (\Delta u)(n) \neq 0, \\ 0, & \text{if } (\Delta u)(n) = 0, \end{cases}$$

it follows that,

$$\begin{aligned}(\Delta(\varphi \circ u))(n) = b_u(n)(\Delta u)(n) &= -b_u(n)\kappa(n)d_1(n) - b_u(n)\kappa(n)\widetilde{\psi}(\widetilde{w}(n)) \\ &= -\widetilde{d_1}(n) - (N \circ \widetilde{w})(n), \quad \forall\, n \in \mathbb{Z}_+, \end{aligned} \tag{5.23}$$

where the function $N : \mathbb{Z}_+ \times \mathbb{R} \to \mathbb{R}$ is defined by

$$N(n, \xi) := b_u(n)\kappa(n)\widetilde{\psi}(\xi), \qquad \forall\, (n, \xi) \in \mathbb{Z}_+ \times \mathbb{R}$$

and $\widetilde{d_1} : \mathbb{Z}_+ \to \mathbb{R}$ is defined by

$$\widetilde{d_1}(n) := b_u(n)\kappa(n)d_1(n), \qquad \forall\, n \in \mathbb{Z}_+.$$

Since φ is non-decreasing and globally Lipschitz with Lipschitz constant λ_1, we see that $0 \le b_u(n) \le \lambda_1$. Combining this with (5.22) yields

$$0 \le N(n, \xi)\xi \le \lambda_1\lambda_2\kappa(n)\xi^2, \qquad \forall\, (n, \xi) \in \mathbb{Z}_+ \times \mathbb{R}. \tag{5.24}$$

It follows from (5.23) that,

$$\varphi \circ u = \varphi(u^0)\vartheta - J\widetilde{d_1} - J(N \circ \widetilde{w}). \tag{5.25}$$

Applying G to both sides of (5.25), and using the fact that by shift-invariance, G commutes with J we obtain

$$G(\varphi \circ u) = \varphi(u^0)G\vartheta - GJ\widetilde{d_1} - J(G(N \circ \widetilde{w})).$$

Invoking (5.21) yields

$$\widetilde{w} = r - J(G(N \circ \widetilde{w})), \tag{5.26}$$

where

$$r := g - \mathbf{G}(1)\varphi^\rho \vartheta + \varphi(u^0)G\vartheta - GJ\widetilde{d}_1.$$

Proof of Statement 1: Clearly, (5.26) is of the form (4.1) with u and φ replaced by $\widetilde{w}$ and N, respectively. Therefore we may apply Corollary 4.2.3, provided the relevant assumptions are satisfied. By (5.20) there exists $a > 0$ satisfying

$$\lambda_1\lambda_2 \limsup_{n\to\infty} \kappa(n) < a < 1/|f_{0,J}(G)|. \tag{5.27}$$

By the definition of $f_{0,J}(G)$, there exists $\varepsilon > 0$ such that

$$\frac{1}{a} + \operatorname{Re}\frac{\mathbf{G}(e^{i\theta})}{e^{i\theta}-1} \geq \varepsilon, \qquad \text{a.a. } \theta \in (0, 2\pi).$$

Moreover, it follows from (5.24) and (5.27) that there exists $n_0 \geq 0$ such that

$$0 \leq N(n,\xi)\xi \leq a\xi^2, \qquad \forall\, n \in [n_0, \infty) \cap \mathbb{Z}_+,\ \forall\, \xi \in \mathbb{R}.$$

The above two inequalities show that (4.53) and (4.58) hold with φ, Q, and P replaced by N, I, and $1/a$, respectively. In order to apply Corollary 4.2.3, it remains to verify that $r \in m^2(\mathbb{Z}_+, \mathbb{R})$. To this end note that

$$r = g + \mathbf{G}(1)(\varphi(u^0)\vartheta - \varphi^\rho\vartheta) + \varphi(u^0)(G\vartheta - \mathbf{G}(1)\vartheta) - GJ\widetilde{d}_1. \tag{5.28}$$

Since b_u, κ are bounded and non-negative, $d_1 \in l^1(\mathbb{Z}_+, \mathbb{R})$ and by assumption $n \mapsto \sum_{j=n}^{\infty} |d_1(j)| \in l^2(\mathbb{Z}_+, \mathbb{R})$, we deduce that $\widetilde{d}_1 = b_u\kappa d_1 \in l^1(\mathbb{Z}_+, \mathbb{R})$ and

$$n \mapsto \sum_{j=n}^{\infty} |\widetilde{d}_1(j)| \in l^2(\mathbb{Z}_+, \mathbb{R}). \tag{5.29}$$

Now since $\widetilde{d}_1 \in l^1(\mathbb{Z}_+, \mathbb{R})$ there exists $\sigma \in \mathbb{R}$ such that,

$$\lim_{n\to\infty} (J\widetilde{d}_1)(n) = \sigma.$$

From (5.29) we deduce that $J\widetilde{d}_1 - \sigma\vartheta \in l^2(\mathbb{Z}_+, \mathbb{R})$. We have,

$$GJ\widetilde{d}_1 = G(J\widetilde{d}_1 - \sigma\vartheta) + \sigma G\vartheta. \tag{5.30}$$

By assumption (A), $\mathbf{G}(1)$ is the l^2-steady-state gain of G and so $(G\vartheta - \mathbf{G}(1)\vartheta) \in l^2(\mathbb{Z}_+, \mathbb{R})$. Consequently, $G\vartheta \in m^2(\mathbb{Z}_+, \mathbb{R})$. Using this, the fact that $G \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ and $J\widetilde{d}_1 - \sigma\vartheta \in l^2(\mathbb{Z}_+, \mathbb{R})$, we deduce from (5.30) that $GJ\widetilde{d}_1 \in$

$m^2(\mathbb{Z}_+, \mathbb{R})$. Furthermore, since $(G\vartheta - \mathbf{G}(1)\vartheta) \in l^2(\mathbb{Z}_+, \mathbb{R})$ and by assumption $g \in l^2(\mathbb{Z}_+, \mathbb{R})$, we conclude from (5.28) that $r \in m^2(\mathbb{Z}_+, \mathbb{R})$. An application of statement 4 of Theorem 4.2.1 to (5.26) combined with (5.23) now yields that

$$\Delta(\varphi \circ u) = -\widetilde{d_1} - (N \circ \widetilde{w}) \in l^2(\mathbb{Z}_+, \mathbb{R}).$$

Furthermore, by the same theorem and (5.25),

$$\lim_{n\to\infty} (\varphi \circ u)(n) = \varphi(u^0) - \sigma - \lim_{n\to\infty} (J(N \circ \widetilde{w}))(n)$$

exists and is finite, completing the proof of statement 1.

Proof of Statement 2: By statement 1, $\Delta(\varphi \circ u) \in l^2(\mathbb{Z}_+, \mathbb{R})$ and $(\varphi \circ u)^\infty = \lim_{n\to\infty}(\varphi \circ u)(n)$ exists and is finite. Hence an application of Proposition 3.4.4 (b) yields,

$$\lim_{n\to\infty} (G(\varphi \circ u))(n) = \mathbf{G}(1)(\varphi \circ u)^\infty.$$

Since $w = g + G(\varphi \circ u)$ and $g \in l^2(\mathbb{Z}_+, \mathbb{R})$ we obtain,

$$\lim_{n\to\infty} w(n) = \mathbf{G}(1)(\varphi \circ u)^\infty.$$

Consequently,

$$y^\infty := \lim_{n\to\infty} y(n) = \lim_{n\to\infty} (\psi \circ w)(n) + d_2 = \psi(\mathbf{G}(1)(\varphi \circ u)^\infty) + d_2,$$

where we have used that $d_1 \in l^1(\mathbb{Z}_+, \mathbb{R})$.

Proof of Statement 3: We know by statement 2 that $y(n) \to y^\infty$ as $n \to \infty$. Seeking a contradiction, suppose that $y^\infty < \rho$ (the case $y^\infty > \rho$ can be treated in an analogous way). Setting $\varepsilon := \rho - y^\infty > 0$ and taking $n_0 \geq 0$ large enough, we have $\rho - y(n) \geq \varepsilon/2$ whenever $n \geq n_0$. Hence we see that,

$$(\Delta u)(n) = \kappa(n)(\rho - y(n)) \geq \kappa(n)\frac{\varepsilon}{2}, \qquad \forall\, n \geq n_0.$$

Summing the above inequality from n_0 to $m-1$ gives

$$u(m) \geq u(n_0) + \frac{\varepsilon}{2}\sum_{k=n_0}^{m-1} \kappa(k) \to \infty, \qquad m \to \infty.$$

Consequently, since φ is non-decreasing,

$$\overline{\varphi} := \sup_{v\in\mathbb{R}} \varphi(v) = (\varphi \circ u)^\infty.$$

Hence, by statement 2, $y^\infty = \psi(\mathbf{G}(1)\overline{\varphi}) + d_2$. Since $\rho - d_2 \in \mathscr{R}(G, \varphi, \psi)$ (by assumption) and using the fact that ψ is non-decreasing and $\mathbf{G}(1) > 0$, we obtain

$$\rho \leq \sup \mathscr{R}(G, \varphi, \psi) + d_2 = \psi(\mathbf{G}(1)\overline{\varphi}) + d_2 = y^\infty,$$

contradicting the supposition that $y^\infty < \rho$. Setting $\underline{\varphi} := \inf_{v \in \mathbb{R}} \varphi(v)$, an analogous argument shows that if $y^\infty > \rho$, then necessarily $y^\infty = \psi(\mathbf{G}(1)\underline{\varphi}) + d_2$, which likewise leads to a contradiction since $\rho \geq \psi(\mathbf{G}(1)\underline{\varphi}) + d_2$.

Proof of Statement 4: By statement 1, the limit $(\varphi \circ u)^\infty = \lim_{n\to\infty}(\varphi \circ u)(n)$ exists and is finite. We consider the following cases:
CASE 1: $\kappa \notin l^1(\mathbb{Z}_+, \mathbb{R})$.
If $\kappa \notin l^1(\mathbb{Z}_+, \mathbb{R})$, then by statements 2 and 3, $\rho - d_2 = \psi(\mathbf{G}(1)(\varphi \circ u)^\infty)$. Unboundedness of u would imply that there exists a sequence (v_n) with $\lim_{n\to\infty} |v_n| = \infty$ and such that,

$$\rho - d_2 = \lim_{n\to\infty} \psi(\mathbf{G}(1)\varphi(v_n)).$$

Since the function $v \mapsto \psi(\mathbf{G}(1)\varphi(v))$ is non-decreasing, this would in turn yield $\rho - d_2 = \sup \mathscr{R}(G, \varphi, \psi)$ or $\rho - d_2 = \inf \mathscr{R}(G, \varphi, \psi)$, showing that u must be bounded if $\rho - d_2$ is an interior point of $\mathscr{R}(G, \varphi, \psi)$.
CASE 2: $\kappa \in l^1(\mathbb{Z}_+, \mathbb{R})$.
By statement 2 we know that $\rho\vartheta - y$ is bounded. With $\kappa \in l^1(\mathbb{Z}_+, \mathbb{R})$ it now follows that $\kappa(\rho\vartheta - y) \in l^1(\mathbb{Z}_+, \mathbb{R})$. Since $\Delta u = \kappa(\rho\vartheta - y)$, we conclude that u is bounded. □

We now change the integrator J to J_0 and consider the feedback system shown in Figure 5.4, where $c \in \mathbb{R}$ is the initial state of the integrator.

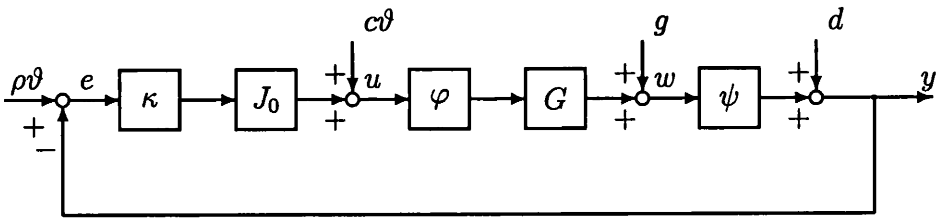

Figure 5.4: Low-gain control problem with time-varying gain and J_0 integrator

From Figure 5.4 we can derive the following governing equations,

$$w = g + G(\varphi \circ u), \qquad y = \psi(w) + d, \qquad u = c\vartheta + J_0(\kappa(\rho\vartheta - y)),$$

or, equivalently

$$u = c\vartheta + J_0(\kappa(\rho\vartheta - d - \psi(g + G(\varphi \circ u)))). \tag{5.31}$$

Note that if $\mathbf{G}(\infty) = 0$, then (5.31) has a unique solution. Since $\Delta_0 J_0 = J_0 \Delta_0 = I$, (5.31) is equivalent to

$$\Delta_0 u = c\delta + \kappa(\rho\vartheta - d - \psi(g + G(\varphi \circ u))). \tag{5.32}$$

We define,

$$f_{0,J_0}(G) := \operatorname{ess\,inf}_{\theta \in (0,2\pi)} \mathrm{Re}\left[\frac{\mathbf{G}(e^{i\theta})e^{i\theta}}{e^{i\theta} - 1}\right]. \tag{5.33}$$

If the transfer function $\mathbf{G}$ of G satisfies assumption (A), then $-\infty < f_{0,J_0}(G) \leq \mathbf{G}(\infty) - \mathbf{G}(1)/2$, where $\mathbf{G}(\infty) := \lim_{|z|\to\infty, z \in \mathbf{E}_1} \mathbf{G}(z)$ (see Appendix 2, Proposition 12.1.4 (ii), for more details).

Theorem 5.2.4. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ be a shift-invariant operator with transfer function $\mathbf{G}$. Assume that assumption* (A) *holds with $\mathbf{G}(1) > 0$, $g \in l^2(\mathbb{Z}_+, \mathbb{R})$, $d = d_1 + d_2\vartheta$ with $d_1 \in l^1(\mathbb{Z}_+, \mathbb{R})$, $d_2 \in \mathbb{R}$ and $n \mapsto \sum_{j=n}^{\infty} |d_1(j)| \in l^2(\mathbb{Z}_+, \mathbb{R})$, $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are non-decreasing and globally Lipschitz continuous with Lipschitz constants $\lambda_1 > 0$ and $\lambda_2 > 0$, $\rho - d_2 \in \mathscr{R}(G, \varphi, \psi)$ and $\kappa : \mathbb{Z}_+ \to \mathbb{R}$ is bounded and non-negative with*

$$\limsup_{n\to\infty} \kappa(n) < 1/|\lambda_1 \lambda_2 f_{0,J_0}(G)|, \tag{5.34}$$

if $f_{0,J_0}(G) \leq 0$, where $1/0 := \infty$. Let $u : \mathbb{Z}_+ \to \mathbb{R}$ be a solution of (5.31). *Then the conclusions of Theorem* 5.2.2 *hold.*

Remark 5.2.5. Note that the range of gains guaranteed to achieve tracking (specified by (5.34)) is potentially larger than in Theorem 5.2.2. In particular, if $f_{0,J_0}(G) > 0$ then condition (5.34) need no longer be imposed on the gain κ. $\diamond$

Proof of Theorem 5.2.4. Let $u : \mathbb{Z}_+ \to \mathbb{R}$ be a solution of (5.31). We shall prove Theorem 5.2.4 by applying Corollary 4.3.5 to the equation satisfied by the input signal

$$w = g + G(\varphi \circ u)$$

of the output non-linearity ψ, modified with an offset which depends on ρ and d_2. We invoke arguments identical to those leading up to (5.22) in the proof of Theorem 5.2.2. Using (5.32),

$$\begin{aligned}
\Delta_0 u &= c\delta + \kappa(\rho\vartheta - d - \psi(g + (G(\varphi \circ u))) \\
&= c\delta + \kappa(\rho\vartheta - d - \psi(\widetilde{w} + \mathbf{G}(1)\varphi^\rho)) \\
&= c\delta - \kappa(d_1 + \widetilde{\psi} \circ \widetilde{w}),
\end{aligned}$$

where $\widetilde{w}$, $\widetilde{\psi}$ are given by (5.21). Defining,

$$b_u(n) := \begin{cases} \frac{(\Delta_0(\varphi\circ u))(n)}{(\Delta_0 u)(n)}, & \text{if } (\Delta_0 u)(n) \neq 0, \\ 0, & \text{if } (\Delta_0 u)(n) = 0, \end{cases}$$

it follows that,

$$\begin{aligned}(\Delta_0(\varphi\circ u))(n) &= b_u(n)(\Delta_0 u)(n) \\ &= cb_u(n)\delta(n) - b_u(n)\kappa(n)d_1(n) - b_u(n)\kappa(n)\widetilde{\psi}(\widetilde{w}(n)) \\ &= cb_u(n)\delta(n) - \widetilde{d}_1(n) - (N\circ\widetilde{w})(n), \quad \forall\, n\in\mathbb{Z}_+,\end{aligned} \tag{5.35}$$

where the function $N : \mathbb{Z}_+\times\mathbb{R}\to\mathbb{R}$ is defined by

$$N(n,\xi) := b_u(n)\kappa(n)\widetilde{\psi}(\xi), \qquad \forall\ (n,\xi)\in\mathbb{Z}_+\times\mathbb{R}$$

and $\widetilde{d}_1 : \mathbb{Z}_+\to\mathbb{R}$ is defined by

$$\widetilde{d}_1(n) := b_u(n)\kappa(n)d_1(n), \qquad \forall\, n\in\mathbb{Z}_+.$$

Since φ is non-decreasing and globally Lipschitz with Lipschitz constant λ_1, we see that $0\le b_u(n)\le\lambda_1$ for all $n\ge 1$. Combining this with (5.22) yields

$$0\le N(n,\xi)\xi\le\lambda_1\lambda_2\kappa(n)\xi^2, \qquad \forall\ (n,\xi)\in\mathbb{N}\times\mathbb{R}. \tag{5.36}$$

It follows from (5.35) that,

$$\varphi\circ u = cb_u(0)\vartheta - J_0\widetilde{d}_1 - J_0(N\circ\widetilde{w}). \tag{5.37}$$

Applying G to both sides of (5.37), and using the fact that by shift-invariance, G commutes with J_0 we obtain

$$G(\varphi\circ u) = cb_u(0)G\vartheta - GJ_0\widetilde{d}_1 - J_0(G(N\circ\widetilde{w})).$$

Invoking (5.21) yields

$$\widetilde{w} = r - J_0(G(N\circ\widetilde{w})), \tag{5.38}$$

where

$$r := g - \mathbf{G}(1)\varphi^\rho\vartheta + cb_u(0)G\vartheta - GJ_0\widetilde{d}_1.$$

Proof of Statement 1: Clearly, (5.38) is of the form (4.106) with u and φ replaced by $\widetilde{w}$ and N, respectively. Therefore we may apply Corollary 4.3.5, provided the relevant assumptions are satisfied. We first assume that $f_{0,J_0}(G)\le 0$. By (5.34) there exists $a > 0$ satisfying

$$\lambda_1 \lambda_2 \limsup_{n \to \infty} \kappa(n) < a < 1/|f_{0,J_0}(G)|. \tag{5.39}$$

By the definition of $f_{0,J_0}(G)$, there exists $\varepsilon > 0$ such that

$$\frac{1}{a} + \operatorname{Re} \frac{\mathbf{G}(e^{i\theta})e^{i\theta}}{e^{i\theta} - 1} \geq \varepsilon, \qquad \text{a.a. } \theta \in (0, 2\pi).$$

Moreover, it follows from (5.36) and (5.39) that there exists $n_0 \geq 1$ such that

$$0 \leq N(n, \xi)\xi \leq a\xi^2, \qquad \forall\, n \in [n_0, \infty) \cap \mathbb{Z}_+,\ \forall\, \xi \in \mathbb{R}.$$

The above two inequalities show that (4.119) and (4.58) hold with φ, Q, and P replaced by N, I, and $1/a$, respectively. Note that if $f_{0,J_0}(G) > 0$, it immediately follows that there exists $\varepsilon > 0$ such that

$$\operatorname{Re} \frac{\mathbf{G}(e^{i\theta})e^{i\theta}}{e^{i\theta} - 1} \geq \varepsilon, \qquad \text{a.a. } \theta \in (0, 2\pi).$$

Moreover, it follows from (5.36) that

$$0 \leq N(n, \xi)\xi \qquad \forall\, (n, \xi) \in \mathbb{N} \times \mathbb{R}.$$

The above two inequalities show that for the case $f_{0,J_0}(G) > 0$, (4.119) and (4.58) hold with φ and Q replaced by N and I, respectively, and $a = \infty$. In order to apply Corollary 4.3.5, it remains to verify that $r \in m^2(\mathbb{Z}_+, \mathbb{R})$. To this end note that

$$r = g + \mathbf{G}(1)(cb_u(0)\vartheta - \varphi^\rho \vartheta) + cb_u(0)(G\vartheta - \mathbf{G}(1)\vartheta) - GJ_0\widetilde{d}_1. \tag{5.40}$$

A similar argument to that in the proof of Theorem 5.2.2 shows $GJ_0\widetilde{d}_1 \in m^2(\mathbb{Z}_+, \mathbb{R})$. By assumption (A), $\mathbf{G}(1)$ is the l^2-steady-state gain of G and so $(G\vartheta - \mathbf{G}(1)\vartheta) \in l^2(\mathbb{Z}_+, \mathbb{R})$. Consequently, since by assumption $g \in l^2(\mathbb{Z}_+, \mathbb{R})$, we conclude from (5.40) that $r \in m^2(\mathbb{Z}_+, \mathbb{R})$. An application of statement 4 of Theorem 4.3.4 to (5.38) combined with (5.35) now yields that

$$\Delta_0(\varphi \circ u) = cb_u\delta - \widetilde{d}_1 - (N \circ \widetilde{w}) \in l^2(\mathbb{Z}_+, \mathbb{R}).$$

Hence it in turn follows that $\Delta(\varphi \circ u) \in l^2(\mathbb{Z}_+, \mathbb{R})$. Now since $\widetilde{d}_1 \in l^1(\mathbb{Z}_+, \mathbb{R})$ there exists $\sigma \in \mathbb{R}$ such that,

$$\lim_{n \to \infty} (J\widetilde{d}_1)(n) = \sigma.$$

Furthermore, by Theorem 4.3.4 and (5.37),

$$\lim_{n\to\infty}(\varphi \circ u)(n) = cb_u(0) - \sigma - \lim_{n\to\infty}\sum_{k=0}^{n}(N \circ \widetilde{w})(k)$$

exists and is finite, completing the proof of statement 1.

The proof of statements 2, 3 and 4 are similar to the proofs of statements 2, 3 and 4 of Theorem 5.2.2 and therefore are omitted. □

5.3 Notes and references

The main results in this chapter, Theorems 5.1.2, 5.1.4, 5.2.2 and 5.2.4, form the basis of §4 of [5]. The aforementioned results are the discrete-time analogues of results in [9]. We emphasize that the main results in this chapter apply to strongly-stable discrete-time state-space systems (see Chapter 6). Previous results on discrete-time integral control, see [17], [34], [37], [42], are obtained only for power stable discrete-time state-space systems using state-space methods. In this Chapter we adopt an input-output approach to the low-gain integral control problem. Note that in [42] results are obtained for power-stable infinite-dimensional state-space systems and in particular, tracking of vector-valued reference signals is considered. However, [42] only treats the linear case not allowing non-linearities in the input or output channel. Moreover, in [34] and [37] only input non-linearities are considered, although in [34] the emphasis is slightly different and the input non-linearity is of hysteresis-type. Whilst the results in [17] allow for input and output non-linearities, these results only apply to finite-dimensional, power-stable state-space systems. We note that Theorems 5.2.2 and 5.2.4 allow for both input and output non-linearities. Note that in the main results of this chapter, tracking of feasible reference values is still achieved in the presence of a large class of output disturbances. Proposition 5.2.1 is new as stated in the input-output setting, but is similar to the discrete-time state-space version contained in [34], (see [34], Proposition 3.2).

Chapter 6

Absolute stability results for infinite-dimensional discrete-time state-space systems with application to low-gain integral control

This chapter is devoted to applications of the results in Chapters 4 and 5 to infinite-dimensional discrete-time state-space systems.

6.1 Power stable and strongly stable discrete-time state-space systems

Consider the discrete-time system

$$x(n+1) = Ax(n) + Bv(n), \qquad x(0) = x^0 \in X, \tag{6.1a}$$
$$w(n) = Cx(n) + Dv(n), \tag{6.1b}$$

with state-space X, input space U, and output space $Y = U$ and generating operators (A, B, C, D). Here X and U are Hilbert spaces, $A \in \mathscr{B}(X)$, $B \in \mathscr{B}(U, X)$, $C \in \mathscr{B}(X, U)$, and $D \in \mathscr{B}(U)$.

We shall use the following notation. We write $\|\cdot\|_{l^p} := \|\cdot\|_{l^p(\mathbf{Z}_+,U)}$ and $\|\cdot\|_{l^p} := \|\cdot\|_{l^p(\mathbf{Z}_+,X)}$ for the l^p-norm of X-valued sequences, where $1 \le p \le \infty$. We denote the spectrum and the spectral radius of A by $\operatorname{spec}(A)$ and $r(A)$, respectively. It is well known that

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

As usual the resolvent set of A is defined by $\operatorname{res}(A) := \mathbb{C} \setminus \operatorname{spec}(A)$.

From (6.1a) we obtain for the solution x of (6.1a),

$$x(n) = \begin{cases} A^n x^0 + \sum_{j=0}^{n-1} A^{(n-1)-j} Bv(j), & n \geq 1, \\ x^0, & n = 0. \end{cases} \tag{6.2}$$

Note that (6.1b) and (6.2) yield the following formula for the input-output operator G:

$$(Gv)(n) = \begin{cases} \sum_{j=0}^{n-1} CA^{(n-1)-j} Bv(j) + Dv(n), & n \geq 1, \\ Dv(0), & n = 0, \end{cases} \quad \forall\, v \in F(\mathbb{Z}_+, U). \tag{6.3}$$

We denote by $\mathbf{G}$ the transfer function of the system (6.1), which, for $|z| > r(A)$, is given by

$$\mathbf{G}(z) = C(zI - A)^{-1}B + D.$$

We say that $A \in \mathscr{B}(X)$ is *power stable* if there exists an $M \geq 1$ and $\gamma \in (0,1)$ such that

$$\|A^n\| \leq M\gamma^n, \qquad \forall\, n \in \mathbb{Z}_+.$$

We say that $A \in \mathscr{B}(X)$ is *strongly stable* if $\lim_{n\to\infty} A^n x = 0$ for all $x \in X$.

The following lemma gives two simple characterisations of power stability.

Lemma 6.1.1. *The following statements are equivalent:*

(i) $A \in \mathscr{B}(X)$ *is power stable;*

(ii) $r(A) < 1$*;*

(iii) $z \mapsto (zI - A)^{-1} \in H^\infty(\mathbb{E}_1, \mathscr{B}(X))$.

A proof of Lemma 6.1.1 can be found in [32] (see, Lemma 1 in [32]).

It is clear that if A is power stable then A is strongly stable, however, the converse is not true as seen from the following example.

Example 6.1.2. Let $X = l^2(\mathbb{Z}_+, \mathbb{R})$ and define $A \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ by

$$A := \operatorname{diag}_{n\in\mathbb{Z}_+}(\lambda_n),$$

where $\lambda_n \in (0,1)$, $n \in \mathbb{Z}_+$, with $\lambda_n \to 1$ as $n \to \infty$. We claim that A is strongly stable, but not power stable. We have,

$$A^k = \operatorname{diag}_{n\in\mathbb{Z}_+}(\lambda_n)^k$$

and $r(A) = 1$. Hence by Lemma 6.1.1, we see that A is not power stable. Let $x \in l^2(\mathbb{Z}_+, \mathbb{R})$. Then,

$$(A^k x)(n) = \lambda_n^k x(n), \quad n \in \mathbb{Z}_+ \quad \text{and} \quad \|A^k x\|_{l^2}^2 = \sum_{n=0}^{\infty} \lambda_n^{2k} x^2(n).$$

Let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that, $\sum_{n=N}^{\infty} x^2(n) \leq \varepsilon/2$ and so,

$$\sum_{n=N}^{\infty} \lambda_n^{2k} x^2(n) \leq \frac{\varepsilon}{2}, \qquad \forall\, k \in \mathbb{N}.$$

Clearly there exists $k_0 > 0$ such that

$$\sum_{n=0}^{N} \lambda_n^{2k} x^2(n) \leq \frac{\varepsilon}{2}, \qquad \forall\, k \geq k_0.$$

Hence,

$$\|A^k x\|_{l^2}^2 = \sum_{n=0}^{\infty} \lambda_n^{2k} x^2(n) \leq \varepsilon, \qquad \forall\, k \geq k_0,$$

showing that A is strongly stable. ◇

We say that system (6.1) is *strongly stable* if the following four conditions are satisfied:

(i) G is l^2-stable, that is, $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$, or, equivalently, the transfer function $\mathbf{G} \in H^\infty(\mathbb{E}_1, \mathscr{B}(U))$.

(ii) A is strongly stable.

(iii) There exists $\alpha \geq 0$ such that,

$$\left\| \sum_{j=0}^{\infty} A^j B v(j) \right\| \leq \alpha \|v\|_{l^2}, \qquad \forall\, v \in l^2(\mathbb{Z}_+, U). \tag{6.4}$$

(iv) There exists $\beta \geq 0$ such that,

$$\left(\sum_{j=0}^{\infty} \|C A^j x\|^2 \right)^{1/2} \leq \beta \|x\|, \qquad \forall\, x \in X. \tag{6.5}$$

The system (6.1) is called *power stable* if A is power stable. It is easy to check that if system (6.1) is power stable, then system (6.1) is strongly stable. It is also clear that the converse is not true (trivial counterexample: A as defined in Example 6.1.2, $B = 0$, $C = 0$ and $D = 0$). Non-trivial counterexamples also exist as seen from the following example.

Example 6.1.3. We give a non-trivial example of a strongly stable system which is not a power stable system. Let $X = l^2(\mathbb{Z}_+, \mathbb{R})$ and let $A \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ be given by

$$A = \operatorname{diag}_{n \in \mathbb{Z}_+}(\lambda_n)$$

for some $\lambda_n \in (0,1)$, $n \in \mathbb{Z}_+$, with $\lambda_n \to 1$ as $n \to \infty$. Let $B \in \mathscr{B}(\mathbb{R}, l^2(\mathbb{Z}_+, \mathbb{R}))$ be given by

$$\xi \mapsto \xi b = \xi \begin{pmatrix} b(0) \\ b(1) \\ \vdots \end{pmatrix}, \qquad \forall\, \xi \in \mathbb{R},$$

where

$$b(n) := (1 - \lambda_n)^{\gamma}(n+1)^{-\delta}, \qquad \forall\, n \in \mathbb{Z}_+,$$

and $\gamma \geq 1/2$ and $\delta > 1/2$. Define $C \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}), \mathbb{R})$ by

$$Cx := \langle b, x \rangle = \begin{pmatrix} b(0) & b(1) & \ldots \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ \vdots \end{pmatrix}, \qquad \forall\, x \in l^2(\mathbb{Z}_+, \mathbb{R}),$$

and set $D = 0$. Then for $z \in \mathbb{E}_1$,

$$(zI - A)^{-1} = \operatorname{diag}_{n \in \mathbb{Z}_+}\left(\frac{1}{z - \lambda_n}\right)$$

and so

$$\mathbf{G}(z) = C(zI - A)^{-1}B + D = \sum_{n=0}^{\infty}\left(\frac{1}{z - \lambda_n} b^2(n)\right).$$

We first show that $\mathbf{G}$ is in $H^{\infty}(\mathbb{E}_1)$. Let $z \in \mathbb{E}_1$. We have,

$$|\mathbf{G}(z)| \leq \sum_{n=0}^{\infty}\left(\frac{1}{|z - \lambda_n|} b^2(n)\right).$$

Note that since $z \in \mathbb{E}_1$ and $\lambda_n \in (0,1)$ for all $n \in \mathbb{Z}_+$,

$$|z - \lambda_n| \geq |z| - \lambda_n > 1 - \lambda_n > 0, \qquad \forall\, n \in \mathbb{Z}_+.$$

Consequently,

$$\frac{1}{|z - \lambda_n|} < \frac{1}{1 - \lambda_n}, \qquad \forall\, n \in \mathbb{Z}_+. \tag{6.6}$$

Hence it follows that

$$|\mathbf{G}(z)| \le \sum_{n=0}^{\infty}\left(\frac{1}{|z-\lambda_n|}b^2(n)\right) < \sum_{n=0}^{\infty}\left(\frac{1}{1-\lambda_n}b^2(n)\right)$$
$$= \sum_{n=0}^{\infty}((1-\lambda_n)^{2\gamma-1}(n+1)^{-2\delta}).$$

Note that since $\gamma \ge 1/2$ and $\lambda_n \in (0,1)$ for all $n \in \mathbb{Z}_+$, the function $n \mapsto (1-\lambda_n)^{2\gamma-1}$ is bounded, taking values in $(0,1]$. Consequently, since $\delta > 1/2$

$$|\mathbf{G}(z)| < \sum_{n=0}^{\infty}(n+1)^{-2\delta} < \infty.$$

Since $z \in \mathbb{E}_1$ was arbitrary it follows that

$$\sup_{z\in\mathbb{E}_1} |\mathbf{G}(z)| < \infty,$$

showing that $\mathbf{G}$ is convergent and bounded for each $z \in \mathbb{E}_1$. It remains to show that $\mathbf{G}$ is holomorphic on $\mathbb{E}_1$. Define $f_n : \mathbb{E}_1 \to \mathbb{C}$ by

$$f_n(z) := \frac{1}{z-\lambda_n}b^2(n), \qquad \forall\, z \in \mathbb{E}_1,\ n \in \mathbb{Z}_+.$$

Using the definition of b and (6.6), it follows that

$$0 \le |f_n(z)| < (1-\lambda_n)^{2\gamma-1}(n+1)^{-2\delta}, \qquad \forall\, z \in \mathbb{E}_1,\ n \in \mathbb{Z}_+.$$

As before, since $\gamma \ge 1/2$ and $\lambda_n \in (0,1)$ for all $n \in \mathbb{Z}_+$, the function $n \mapsto (1-\lambda_n)^{2\gamma-1}$ is bounded, taking values in $(0,1]$. Consequently, since $\delta > 1/2$ it follows that

$$\sum_{n=0}^{\infty}(1-\lambda_n)^{2\gamma-1}(n+1)^{-2\delta} < \infty.$$

An application of the Weierstrass M-test now shows that

$$\sum_{n=0}^{\infty} f_n(z) = \sum_{n=0}^{\infty}\frac{1}{z-\lambda_n}b^2(n) = \mathbf{G}(z),$$

converges uniformly on $\mathbb{E}_1$ to $\mathbf{G}$. Noting that for each $n \in \mathbb{Z}_+$, f_n is holomorphic on $\mathbb{E}_1$, it follows that the uniform limit of the f_n's, that is $\mathbf{G}$, is holomorphic on $\mathbb{E}_1$. Combining this with the fact that $\mathbf{G}$ is bounded on $\mathbb{E}_1$ we see that $\mathbf{G} \in H^\infty(\mathbb{E}_1)$.

Note that it follows from Example 6.1.2 that A is strongly stable, but not power stable.

To see that (6.4) holds, we observe the following. Let $v \in l^2(\mathbb{Z}_+, \mathbb{R})$. Then,

$$\left\| \sum_{j=0}^{\infty} A^j Bv(j) \right\|_X^2 = \sum_{k=0}^{\infty} \left| \left(\sum_{j=0}^{\infty} \lambda_k^j b(k) v(j) \right) \right|^2 \leq \sum_{k=0}^{\infty} |b(k)|^2 \left(\sum_{j=0}^{\infty} |\lambda_k^j v(j)| \right)^2. \tag{6.7}$$

Note that (for fixed k) by Hölders inequality,

$$\sum_{j=0}^{\infty} |\lambda_k^j v(j)| \leq \left(\sum_{j=0}^{\infty} |\lambda_k^j|^2 \right)^{1/2} \left(\sum_{j=0}^{\infty} |v(j)|^2 \right)^{1/2} = \left(\frac{1}{1-\lambda_k^2} \right)^{1/2} \|v\|_X. \tag{6.8}$$

Combining (6.7) and (6.8) we obtain,

$$\left\| \sum_{j=0}^{\infty} A^j Bv(j) \right\|_X^2 \leq \sum_{k=0}^{\infty} \frac{b^2(k)}{1-\lambda_k^2} \|v\|_X^2. \tag{6.9}$$

Using the definition of b, it follows that

$$\sum_{k=0}^{\infty} \frac{b^2(k)}{1-\lambda_k^2} = \sum_{k=0}^{\infty} \frac{b^2(k)}{(1-\lambda_k)(1+\lambda_k)} = \sum_{k=0}^{\infty} (1-\lambda_k)^{2\gamma-1} (1+\lambda_k)^{-1} (k+1)^{-2\delta}.$$

Since $\gamma \geq 1/2$ and $\lambda_k \in (0,1)$ for all $k \in \mathbb{Z}_+$, it follows that $(1-\lambda_k)^{2\gamma-1}/(1+\lambda_k) \in (0,1)$. Consequently, noting that $\delta > 1/2$,

$$\sum_{k=0}^{\infty} \frac{b^2(k)}{1-\lambda_k^2} < \sum_{k=0}^{\infty} (k+1)^{-2\delta} < \infty. \tag{6.10}$$

Combining (6.9) and (6.10) we obtain,

$$\left\| \sum_{j=0}^{\infty} A^j Bv(j) \right\|_X \leq \alpha \|v\|_X = \alpha \|v\|_{l^2}$$

with $\alpha = (\sum_{k=0}^{\infty} (k+1)^{-2\delta})^{1/2}$. Hence, (6.4) holds.

Finally we show that (6.5) holds. Let $x \in X = l^2(\mathbb{Z}_+, \mathbb{R})$. Then,

$$\sum_{j=0}^{\infty} |CA^j x|^2 = \sum_{j=0}^{\infty} \left| \sum_{k=0}^{\infty} \lambda_k^j b(k) x(k) \right|^2 \leq \sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} |\lambda_k^j b(k) x(k)| \right)^2. \tag{6.11}$$

Again (for fixed j) by Hölders inequality,

$$\sum_{k=0}^{\infty} |\lambda_k^j b(k) x(k)| \leq \left(\sum_{k=0}^{\infty} |\lambda_k^j b(k)|^2 \right)^{1/2} \|x\|_{l^2} = \left(\sum_{k=0}^{\infty} |\lambda_k^j b(k)|^2 \right)^{1/2} \|x\|_X.$$

Combining this with (6.11) yields

$$\sum_{j=0}^{\infty} |CA^j x|^2 \le \sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} |\lambda_k^j b(k)|^2 \right) \|x\|_X^2. \tag{6.12}$$

Note that

$$\sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} |\lambda_k^j b(k)|^2 \right) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \lambda_k^{2j} b^2(k) = \sum_{k=0}^{\infty} \frac{b^2(k)}{1-\lambda_k^2}.$$

Using this in (6.12), it follows from (6.10) that

$$\left(\sum_{j=0}^{\infty} |CA^j x|^2 \right)^{1/2} \le \left(\sum_{k=0}^{\infty} \frac{b^2(k)}{1-\lambda_k^2} \right)^{1/2} \|x\|_X \le \beta \|x\|_X,$$

with $\beta = \alpha \ge 0$. Hence, (6.5) holds.

With (A, B, C, D) as defined above we have seen that system (6.1) is a strongly stable discrete-time system but not a power stable system. $\diamond$

Remark 6.1.4. Let $\Lambda := \{\lambda_n\}_{n\in\mathbb{Z}_+} \cup \{1\}$, where the $\{\lambda_n\}_{n\in\mathbb{Z}_+}$ are as in Example 6.1.3. We remark that the transfer function $\mathbf{G}$ constructed in Example 6.1.3 is holomorphic on $\mathbb{C} \setminus \Lambda$. To see this, let $K \subset \mathbb{C} \setminus \Lambda$ be non-empty and compact. Then there exists $\varepsilon > 0$ such that $\operatorname{dist}(K, \Lambda) > \varepsilon$. Consequently, for $z \in K$ we have $|z - \lambda_n| > \varepsilon$. Hence it follows that $|\mathbf{G}(z)| < (1/\varepsilon) \sum_{n=0}^{\infty} b^2(n)$. Arguments similar to those in Example 6.1.3 show that $\mathbf{G}$ is holomorphic on K. Hence, we deduce that $\mathbf{G}$ is holomorphic on $\mathbb{C} \setminus \Lambda$. Moreover, $\mathbf{G}$ has poles at each λ_n, $n \in \mathbb{Z}_+$, an essential singularity at the point 1 and is uniformly bounded on the exterior of the closed unit disc. $\diamond$

To obtain state-space versions of the results in Chapters 4 and 5, we require the following lemmas. The proofs of the following three results can be found in Appendix 5.

Lemma 6.1.5. *Assume that A is strongly stable and* (6.4) *holds. Then there exists $K \ge 0$ such that, for all $x^0 \in X$ and $v \in l^2(\mathbb{Z}_+, U)$, the solution x of* (6.1a) *satisfies*

$$\|x\|_{l^\infty} \le K(\|x^0\| + \|v\|_{l^2}).$$

Moreover, $\lim_{n\to\infty} x(n) = 0$.

Lemma 6.1.6. *Assume that A is strongly stable, $1 \in \operatorname{res}(A)$ and* (6.4) *holds. Let $v \in F(\mathbb{Z}_+, U)$ be such that $\Delta v \in l^2(\mathbb{Z}_+, U)$. Then for all $x^0 \in X$, the solution x of* (6.1a) *satisfies*

$$\lim_{n\to\infty} (x(n) - (I - A)^{-1} B v(n)) = 0.$$

Lemma 6.1.7. *Assume that A is power stable. Then there exists $K \geq 0$ such that, for all $x^0 \in X$ and $v \in l^\infty(\mathbb{Z}_+, U)$, the solution x of* (6.1a) *satisfies*

$$\|x\|_{l^\infty} \leq K(\|x^0\| + \|v\|_{l^\infty}).$$

Moreover, if $\lim_{n\to\infty} v(n) =: v^\infty$ exists then

$$\lim_{n\to\infty} x(n) = (I - A)^{-1} B v^\infty.$$

6.2 Absolute stability results for discrete-time state-space systems

This section is devoted to applications of the results in Chapter 4 to discrete-time state-space systems.

Let $\varphi : \mathbb{Z}_+ \times U \to U$ be a (time-dependent) static non-linearity and consider system (6.1), with input non-linearity $v = \varphi \circ u$ (where, slightly abusing notation, $\varphi \circ u$ denotes the function $n \mapsto \varphi(n, u(n))$), in feedback interconnection with the integrator $\Delta u = -w$, that is,

$$x(n+1) = Ax(n) + B(\varphi \circ u)(n), \qquad x(0) = x^0 \in X, \tag{6.13a}$$

$$u(n+1) = u(n) - Cx(n) - D(\varphi \circ u)(n), \qquad u(0) = u^0 \in U. \tag{6.13b}$$

Then it is clear by iterating (6.13b), that

$$u = u^0 \vartheta - J(Cx + D(\varphi \circ u)). \tag{6.14}$$

We have the following lemma.

Lemma 6.2.1. *Define $r \in F(\mathbb{Z}_+, U)$ by*

$$r(n) := \begin{cases} u^0 - \sum_{j=0}^{n-1} CA^j x^0, & n \geq 1, \\ u^0, & n = 0. \end{cases} \tag{6.15}$$

Then for u given by (6.13b),

$$u = r - J(G(\varphi \circ u)). \tag{6.16}$$

Proof. We have by (6.2) and (6.3),

$$\begin{aligned}
&Cx(n) + D(\varphi \circ u)(n) \\
&= \begin{cases} CA^n x^0 + \sum_{j=0}^{n-1} CA^{(n-1)-j} B(\varphi \circ u)(j) + D(\varphi \circ u)(n), & n \geq 1, \\ Cx^0 + D(\varphi \circ u)(0), & n = 0, \end{cases} \\
&= CA^n x^0 + (G(\varphi \circ u))(n), \qquad \forall\, n \in \mathbb{Z}_+.
\end{aligned}$$

Therefore by (6.14),

$$u(n) = \begin{cases} u^0 - \sum_{j=0}^{n-1} [CA^j x^0 - (G(\varphi \circ u))(j)], & n \geq 1, \\ u^0, & n = 0. \end{cases}$$

Hence with r as defined in (6.15), it is clear that (6.16) holds. □

Remark 6.2.2. Before stating the main results of this section, we remark that if system (6.1) is strongly stable, then $\mathbf{G} \in H^\infty(\mathbb{E}_1, \mathscr{B}(U))$ and hence $\mathbf{G}$ is analytic on $\mathbb{E}_1$. If additionally $1 \in \operatorname{res}(A)$, then $\mathbf{G}$ extends analytically to a neighbourhood of 1, so that

$$\lim_{z \to 1} \frac{1}{z-1}(\mathbf{G}(z) - \mathbf{G}(1)) = \mathbf{G}'(1) = -C(I - A)^{-2} B,$$

and thus

$$\lim_{z \to 1} \frac{\mathbf{G}(z) - \mathbf{G}(1) - (z-1)\mathbf{G}'(1)}{(z-1)^2} = \frac{1}{2}\mathbf{G}''(1),$$

showing that $\mathbf{G}$ satisfies assumption (A′) and hence, in particular, assumption (A). ◇

In the next two results we assume that φ is time-independent and that $U = \mathbb{R}$.

Theorem 6.2.3. *Assume that system* (6.1) *is strongly stable,* $1 \in \operatorname{res}(A)$ *and* $\mathbf{G}(1) > 0$. *Let* $\varphi : \mathbb{R} \to \mathbb{R}$ *be a measurable non-decreasing non-linearity and let* (x, u) *be the unique solution of* (6.13). *If there exist numbers* $q \geq 0$, $\varepsilon > 0$ *and* $a \in (0, \infty)$ *such that* (4.2) *and* (4.3) *hold, then there exists a constant* $K \geq 0$ *(which depends only on* (A, B, C, D), q, ε, a *and* G, *but not on* u^0 *and* x^0) *such that*

$$\|x\|_{l^\infty} + \|u\|_{l^\infty} + \|\Delta_0 u\|_{l^2} + \|\varphi \circ u\|_{l^2} \leq K(\|x^0\| + |u^0|).$$

Moreover, $\lim_{n \to \infty} x(n) = 0$.

The proof of Theorem 6.2.3 follows from an application of Theorem 4.1.2. The arguments used to prove Theorem 6.2.3 are similar to those used to prove Theorem 6.2.5 and so we omit the proof of Theorem 6.2.3.

Theorem 6.2.4. *Assume that system* (6.1) *is power stable and* $\mathbf{G}(1) > 0$. *Let* $\varphi : \mathbb{R} \to \mathbb{R}$ *be measurable, locally Lipschitz and non-decreasing and let* (x, u) *be the unique solution of* (6.13). *If there exist numbers* $q \geq 0$ *and* $a \in (0, \infty)$ *such that* (4.2) *and* (4.40) *hold, then, for each* $R > 0$, *there exists a constant* $K_R \geq 0$ *(which depends only on* R, (A, B, C, D), q, a *and* G, *but not on* u^0 *and* x^0*) such that*

$$\|x\|_{l^\infty} + \|u\|_{l^\infty} \leq K_R(\|x^0\| + |u^0|), \tag{6.17}$$

for all $(x^0, u^0) \in X \times \mathbb{R}$ *with* $\|x^0\| + |u^0| \leq R$.

The proof of Theorem 6.2.4 follows from an application of Theorem 4.1.3. The arguments used to prove Theorem 6.2.4 are similar to those used to prove Theorem 6.2.6 and so we omit the proof of Theorem 6.2.4.

Theorem 6.2.5. *Assume that system* (6.1) *is strongly stable,* $1 \in \operatorname{res}(A)$ *and* $\mathbf{G}(1)$ *is invertible. Let* $\varphi : \mathbb{Z}_+ \times U \to U$ *be a non-linearity. Suppose there exist self-adjoint* $P \in \mathscr{B}(U)$, *invertible* $Q \in \mathscr{B}(U)$ *with* $Q\mathbf{G}(1) = [Q\mathbf{G}(1)]^* \geq 0$ *and a number* $\varepsilon > 0$ *such that* (4.52) *and* (4.53) *hold. Let* (x, u) *be the unique solution of* (6.13). *Then there exists a constant* $K \geq 0$ *(which depends only on* (A, B, C, D), G, Q, P *and* ε, *but not on* u^0 *and* x^0*) such that*

$$\|x\|_{l^\infty} + \|u\|_{l^\infty} + \|\Delta u\|_{l^2} + \|\varphi \circ u\|_{l^2} \leq K(\|x^0\| + \|u^0\|). \tag{6.18}$$

Moreover, $\lim_{n\to\infty} x(n) = 0$.

Proof. Let (x, u) be the unique solution of (6.13). By Lemma 6.2.1, u satisfies (6.16) with r given by (6.15). In order to apply Theorem 4.2.1 to (6.16), we need to verify the relevant assumptions. It is clear (see Remark 6.2.2), that $\mathbf{G}$ satisfies assumption (A). To show that $r \in m^2(\mathbb{Z}_+, U)$, we first note that,

$$\sum_{j=0}^{n-1} CA^j x^0 = C(I - A)^{-1}x^0 - CA^n(I - A)^{-1}x^0, \qquad \forall\, n \geq 1.$$

Hence

$$r(n) = u^0 - CA(I - A)^{-1}x^0 + CA^n(I - A)^{-1}x^0, \qquad \forall\, n \in \mathbb{Z}_+.$$

We remark that since $1 \in \operatorname{res}(A)$, we have that $(I - A)^{-1} \in \mathscr{B}(X)$. By (6.5), the function $n \mapsto CA^n(I - A)^{-1}x^0 \in l^2(\mathbb{Z}_+, U)$. Consequently, $r \in m^2(\mathbb{Z}_+, U)$.

An application of Theorem 4.2.1 shows that there exists a constant $K_1 \geq 0$ (not depending on r) such that

$$\|u\|_{l^\infty} + \|\Delta u\|_{l^2} + \|\varphi \circ u\|_{l^2} \leq K_1\|r\|_{m^2}. \tag{6.19}$$

Since $\varphi \circ u \in l^2(\mathbb{Z}_+, U)$, it follows from Lemma 6.1.5 that $x \in l^\infty(\mathbb{Z}_+, U)$, $\lim_{n\to\infty} x(n) = 0$ and

$$\|x\|_{l^\infty} \leq K_2(\|x^0\| + \|\varphi \circ u\|_{l^2}), \tag{6.20}$$

for some suitable constant $K_2 \geq 0$ (not depending on x^0 and u).

Using (6.5) we have

$$\begin{aligned} \|r\|_{m^2} &= \left(\sum_{j=0}^{\infty} \|CA^j(I-A)^{-1}x^0\|^2 \right)^{1/2} + \|u^0 - CA(I-A)^{-1}x^0\| \\ &\leq \beta\|(I-A)^{-1}x^0\| + \|u^0 - CA(I-A)^{-1}x^0\|, \end{aligned}$$

for some $\beta \geq 0$. Therefore,

$$\|r\|_{m^2} \leq K_3(\|x^0\| + \|u^0\|) \tag{6.21}$$

for suitable constant $K_3 \geq 0$ (not depending on x^0 and u^0). Combining (6.19)-(6.21) shows that there exists a constant $K \geq 0$ (not depending on x^0 and u^0), such that (6.18) holds. □

If $\varphi(n, 0) = 0$ for all $n \in \mathbb{Z}_+$, then for $x^0 = 0$ and $u^0 = 0$, the trivial function $n \mapsto (0, 0)$ is the unique solution of (6.13), called the *zero solution.*

Theorem 6.2.6. *Assume that system* (6.1) *is power stable and* $\mathbf{G}(1)$ *is invertible. Let* $\varphi : \mathbb{Z}_+ \times U \to U$ *be a non-linearity and* (x, u) *be the unique solution of* (6.13). *If there exist self-adjoint* $P \in \mathscr{B}(U)$, *invertible* $Q \in \mathscr{B}(U)$ *with* $Q\mathbf{G}(1) = [Q\mathbf{G}(1)]^* \geq 0$ *such that* (4.52) *and* (4.92) *hold, then the following statements hold.*

1. *We have* $x \in l^\infty(\mathbb{Z}_+, X)$ *and* $u \in l^\infty(\mathbb{Z}_+, U)$. *Furthermore, assume that* $\varphi : \mathbb{Z}_+ \times U \to U$ *is such that* $(n, v) \mapsto \varphi(n, v)$ *is locally Lipschitz in* v, *uniformly in* n. *Then, for each* $R > 0$, *there exists a constant* $K_R \geq 0$ (*which depends only on* R, (A, B, C, D), G, Q *and* P, *but not on* u^0 *and* x^0) *such that*
$$\|x\|_{l^\infty} + \|u\|_{l^\infty} \leq K_R(\|x^0\| + \|u^0\|), \tag{6.22}$$
for all $(x^0, u^0) \in X \times U$ *with* $\|x^0\| + \|u^0\| \leq R$.

2. *If* φ *satisfies assumptions* (B), (C) (*see Theorem* 4.2.1) *and* (E) (*see Theorem* 4.2.4) *and, additionally,* φ *is continuous and* $\varphi^{-1}(0)$ *is totally disconnected, then*
$$\lim_{n\to\infty} u(n) \in \varphi^{-1}(0) \quad \text{and} \quad \lim_{n\to\infty} x(n) = 0.$$

If further $\varphi^{-1}(0) = \{0\}$*, then the zero solution of* (6.13) *is globally attractive, that is*

$$\lim_{n\to\infty} u(n) = 0 \quad \textit{and} \quad \lim_{n\to\infty} x(n) = 0,$$

for all $(x^0, u^0) \in X \times U$.

Proof. Let (x, u) be the unique solution of (6.13). By Lemma 6.2.1, u satisfies (6.16) with r given by (6.15). In order to apply Theorem 4.2.4 to (6.16), we need to verify the relevant assumptions. Since system (6.1) is power stable we have $\mathbf{G} \in H^\infty(\mathbb{E}_\alpha, \mathscr{B}(U))$ for some $\alpha \in (0,1)$ and hence $\mathbf{G}$ satisfies assumption (A). To see that $r \in m^1(\mathbb{Z}_+, U)$, note that, as in the proof of Theorem 6.2.5, we have

$$r(n) = u^0 - C(I-A)^{-1}x^0 + CA^n(I-A)^{-1}x^0, \qquad \forall\, n \in \mathbb{Z}_+. \tag{6.23}$$

It follows from the power stability of A that $n \mapsto CA^n(I-A)^{-1}x^0$ is in $l^1(\mathbb{Z}_+, U)$ and hence $r \in m^1(\mathbb{Z}_+, U)$.

Proof of statement 1: An application of Theorem 4.2.4 now yields $u \in l^\infty(\mathbb{Z}_+, U)$ and that there exists a constant $K_1 \geq 0$ (not depending on r), such that

$$\|u\|_{l^\infty} \leq K_1 \|r\|_{m^1}. \tag{6.24}$$

Note that by power stability of A, it follows from (6.23) that

$$\|r\|_{m^1} \leq K_2(\|x^0\| + \|u^0\|) \tag{6.25}$$

for some constant $K_2 \geq 0$ (not depending on x^0 or u^0). Combining (6.24) and (6.25) yields

$$\|u\|_{l^\infty} \leq K_3(\|x^0\| + \|u^0\|), \tag{6.26}$$

for some constant $K_3 \geq 0$ (not depending on x^0 or u^0).

Since $u \in l^\infty(\mathbb{Z}_+, U)$ and φ is locally Lipschitz, it follows that $\varphi \circ u \in l^\infty(\mathbb{Z}_+, U)$. Consequently, by Lemma 6.1.7, $x \in l^\infty(\mathbb{Z}_+, X)$ and there exists a constant $K_4 \geq 0$ (not depending on x^0 and n) such that

$$\|x\|_{l^\infty} \leq K_4(\|x^0\| + \|\varphi \circ u\|_{l^\infty}). \tag{6.27}$$

Let $R > 0$ and assume that $\|x^0\| + \|u^0\| \leq R$. By (4.52), $\varphi(n, 0) = 0$ (that is, φ is unbiased). Combining the fact that φ is locally Lipschitz with its unbiasedness, there exists a constant $K_5 \geq 0$ such that

$$\|\varphi(n, \xi)\| \leq K_5 \|\xi\|, \qquad \forall\, n \in \mathbb{Z}_+, \quad \|\xi\| < R.$$

Consequently, by (6.26), there exists a constant $K_6 \geq 0$ (not depending on x^0 or u^0) such that

$$\|(\varphi \circ u)(n)\| \leq K_6(\|x^0\| + \|u^0\|). \tag{6.28}$$

By (6.28), together with (6.27) and (6.26), it follows that there exists a constant $K_R \geq 0$ such that (6.22) holds.

Proof of Statement 2: If φ satisfies assumptions (B), (C) and (E), φ is continuous and $\varphi^{-1}(0)$ is totally disconnected, an application of Theorem 4.2.4 shows that $\lim_{n\to\infty} u(n) \in \varphi^{-1}(0)$. Therefore, $\lim_{n\to\infty}(\varphi \circ u)(n) = 0$ and so by Lemma 6.1.7 we see that $\lim_{n\to\infty} x(n) = 0$. It is clear that if $\varphi^{-1}(0) = \{0\}$, then $\lim_{n\to\infty} u(n) = 0$ and the last part of statement 2 follows. □

We now consider system (6.1) with input non-linearity $v = \varphi \circ u$ in feedback interconnection with the integrator

$$\begin{aligned} z(n+1) &= z(n) - Cx(n) - D(\varphi \circ u)(n), \quad z(0) = z^0 \in U, \\ u(n) &= z(n) - Cx(n) - D(\varphi \circ u)(n), \end{aligned}$$

where z denotes the integrator state. Equivalently,

$$x(n+1) = Ax(n) + B(\varphi \circ u)(n), \quad x(0) = x^0 \in X, \tag{6.29a}$$
$$z(n+1) = z(n) - Cx(n) - D(\varphi \circ u)(n), \quad z(0) = z^0 \in U, \tag{6.29b}$$
$$u(n) = z(n) - Cx(n) - D(\varphi \circ u)(n), \quad u(0) = u^0 \in U. \tag{6.29c}$$

Then it is clear by (6.29b) and (6.29c), that,

$$u = z^0\vartheta - J_0(Cx + D(\varphi \circ u)). \tag{6.30}$$

It can easily be seen that (6.29) has at least one solution (x, u) (a unique solution, respectively) if, for every $n \in \mathbb{Z}_+$, the map $f_n : U \to U$ defined by

$$f_n(\xi) := \xi + D\varphi(n, \xi), \quad \forall\, (n, \xi) \in \mathbb{Z}_+ \times U,$$

is surjective (bijective, respectively).

We have the following lemma.

Lemma 6.2.7. *Define $r \in F(\mathbb{Z}_+, U)$ by*

$$r(n) := z^0 - \sum_{j=0}^{n} CA^j x^0, \qquad \forall\, n \in \mathbb{Z}_+.$$

Then,

$$u = r - J_0(G(\varphi \circ u)). \tag{6.31}$$

Proof. The proof of Lemma 6.2.7 is similar to the proof of Lemma 6.2.1. □

In the next two results we assume that φ is time-independent and that $U = \mathbb{R}$.

Theorem 6.2.8. *Assume that system* (6.1) *is strongly stable,* $1 \in \operatorname{res}(A)$ *and* $\mathbf{G}(1) > 0$. *Let* $\varphi : \mathbb{R} \to \mathbb{R}$ *be a measurable non-decreasing non-linearity and let* (x, u) *be a solution of* (6.29). *If there exist numbers* $q \geq 0$, $\varepsilon > 0$ *and* $a \in (0, \infty]$ *such that* (4.2) *and* (4.107) *hold then the conclusions of Theorem* 6.2.3 *hold.*

Proof. The proof of Theorem 6.2.8 is similar to the proof of Theorem 6.2.3, but in this case an application of Theorem 4.3.2 to (6.31) is required. □

Theorem 6.2.9. *Assume that system* (6.1) *is power stable and* $\mathbf{G}(1) > 0$. *Let* $\varphi : \mathbb{R} \to \mathbb{R}$ *be measurable, locally Lipschitz and non-decreasing. Suppose there exist numbers* $q \geq 0$ *and* $a \in (0, \infty]$ *such that* (4.2) *and* (4.118) *hold. Let* (x, u) *be a solution of* (6.29). *Then the conclusions of Theorem* 6.2.4 *hold.*

Proof. The proof of Theorem 6.2.9 is similar to the proof of Theorem 6.2.4, but in this case an application of Theorem 4.3.3 to (6.31) is required. □

Theorem 6.2.10. *Assume that system* (6.1) *is strongly stable,* $1 \in \operatorname{res}(A)$ *and* $\mathbf{G}(1)$ *is invertible. Let* $\varphi : \mathbb{Z}_+ \times U \to U$ *be a non-linearity. Suppose there exist self-adjoint* $P \in \mathscr{B}(U)$, *invertible* $Q \in \mathscr{B}(U)$ *with* $Q\mathbf{G}(1) = [Q\mathbf{G}(1)]^* \geq 0$ *and a number* $\varepsilon > 0$ *such that* (4.52) *and* (4.119) *hold. Let* (x, u) *be a solution of* (6.29). *Then the conclusions of Theorem* 6.2.5 *hold.*

Proof. The proof of Theorem 6.2.10 is similar to the proof of Theorem 6.2.5, but in this case an application of Theorem 4.3.4 to (6.31) is required. □

Theorem 6.2.11. *Assume that system* (6.1) *is power stable and* $\mathbf{G}(1)$ *is invertible. Let* $\varphi : \mathbb{Z}_+ \times U \to U$ *be a non-linearity. Suppose there exist self-adjoint* $P \in \mathscr{B}(U)$, *invertible* $Q \in \mathscr{B}(U)$ *with* $Q\mathbf{G}(1) = [Q\mathbf{G}(1)]^* \geq 0$ *such that,* (4.52) *and* (4.131) *hold. Let* (x, u) *be a solution of* (6.29). *Then the conclusions of Theorem* 6.2.6 *hold.*

Proof. The proof of Theorem 6.2.11 is similar to the proof of Theorem 6.2.6, but in this case an application of Theorem 4.3.6 to (6.31) is required. □

6.3 Low-gain integral control of discrete-time state-space systems subject to input/output non-linearities

In this section we present 'state-space' versions of the results in Chapter 5. We assume throughout this section that $U = \mathbb{R}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a static non-linearity, $\rho, k \in \mathbb{R}$ and d be an external disturbance. Consider the discrete-time system (6.1), with input non-linearity $v = \varphi \circ u$ in feedback interconnection with the integrator $\Delta u = k(\rho\vartheta - d - w)$, that is,

$$x(n+1) = Ax(n) + B(\varphi \circ u)(n), \quad x(0) = x^0 \in X, \tag{6.32a}$$
$$u(n+1) = u(n) + k(\rho - d(n) - Cx(n) - D(\varphi \circ u)(n)), \quad u(0) = u^0 \in \mathbb{R}. \tag{6.32b}$$

Then it is clear by (6.32b) that

$$u = u^0\vartheta + kJ(\rho\vartheta - d - Cx - D(\varphi \circ u)). \tag{6.33}$$

We now have the following lemma.

Lemma 6.3.1. *Define $g \in F(\mathbb{Z}_+, \mathbb{R})$ by $g(n) := CA^n x^0$ for all $n \in \mathbb{Z}_+$. Then for u given by* (6.32b),

$$u = u^0\vartheta + kJ(\rho\vartheta - (g + d + G(\varphi \circ u))). \tag{6.34}$$

Proof. With $g(n) := CA^n x^0$ we have by (6.2) and (6.3),

$$\begin{aligned}
&Cx(n) + D(\varphi \circ u)(n) \\
&= \begin{cases} CA^n x^0 + \sum_{j=0}^{n-1} CA^{(n-1)-j} B(\varphi \circ u)(j) + D(\varphi \circ u)(n), & n \geq 1, \\ Cx^0 + D(\varphi \circ u)(0), & n = 0, \end{cases} \\
&= CA^n x^0 + (G(\varphi \circ u))(n) \\
&= g(n) + (G(\varphi \circ u))(n), \qquad \forall\, n \in \mathbb{Z}_+.
\end{aligned}$$

Therefore by (6.33),

$$u(n) = \begin{cases} u^0 + k\sum_{j=0}^{n-1}[\rho - (g(j) + d(j) + (G(\varphi \circ u))(j)], & n \geq 1, \\ u^0, & n = 0, \end{cases}$$

or equivalently, (6.34) holds. □

Theorem 6.3.2. *Assume that system* (6.1) *is strongly stable,* $1 \in \operatorname{res}(A)$ *and* $\mathbf{G}(1) > 0$. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous and non-decreasing. Suppose that $d = d_1 + d_2\vartheta$ with $Jd_1 \in m^2(\mathbb{Z}_+, \mathbb{R})$ and $d_2 \in \mathbb{R}$. Let $\rho \in \mathbb{R}$, assume that $(\rho - d_2)/\mathbf{G}(1) \in \operatorname{im}\varphi$ and let (x, u) be the unique solution of* (6.32). *Under these conditions the following statements hold.*

1. *Assume that $\varphi - \varphi(0) \in \mathscr{S}(a)$ for some $a \in (0, \infty)$. Then there exists a constant $k^* \in (0, \infty)$ (depending on G, φ and ρ) such that for all $k \in (0, k^*)$, the limits $\lim_{n\to\infty} x(n) =: x^\infty$ and $\lim_{n\to\infty} u(n) =: u^\infty$ exist and satisfy*

$x^\infty = ((\rho - d_2)/\mathbf{G}(1))(I-A)^{-1}B$ *and* $\varphi(u^\infty) = (\rho - d_2)/\mathbf{G}(1)$,

$$e = \Delta u/k \in l^2(\mathbb{Z}_+,\mathbb{R}) \quad \textit{and} \quad \varphi\circ u - \varphi(u^\infty) \in l^2(\mathbb{Z}_+,\mathbb{R}).$$

In particular, $\lim_{n\to\infty} e(n) = 0$.

2. *Assume that* φ *is globally Lipschitz continuous with Lipschitz constant* $\lambda > 0$. *Then the conclusions of statement 1 are valid with* $k^* = 1/|\lambda f_J(G)|$, *where* $f_J(G)$ *is given by* (5.2).

Proof. By Lemma 6.3.1, u satisfies

$$u = u^0\vartheta + kJ(\rho\vartheta - (g + d + G(\varphi\circ u))). \tag{6.35}$$

where $g(n) := CA^n x^0$.

Proof of Statement 1: In order to apply Theorem 5.1.2 to (6.35), we need to verify the relevant assumptions. It is clear (see Remark 6.2.2) that $\mathbf{G}$ satisfies assumption (A′). By an argument identical to that in the proof of Theorem 6.2.5, the function $n \mapsto (Jg)(n)$ is in $m^2(\mathbb{Z}_+,\mathbb{R})$.

An application of Theorem 5.1.2 shows that there exists $k^* > 0$ (not depending on x^0 and u^0) such that if $k \in (0,k^*)$ then $\lim_{n\to\infty} u(n) =: u^\infty$ exists and satisfies $\varphi(u^\infty) = (\rho - d_2)/\mathbf{G}(1)$, $e \in l^2(\mathbb{Z}_+,\mathbb{R})$ and $\varphi\circ u - \varphi(u^\infty) \in l^2(\mathbb{Z}_+,\mathbb{R})$. For the rest of the proof of statement 1, let $k \in (0,k^*)$. To prove the remaining assertions in statement 1, let $u^\rho \in \mathbb{R}$ be such that $\varphi(u^\rho) = (\rho - d_2)/\mathbf{G}(1)$. Define $z(\cdot) := x(\cdot) - x^\infty$ and $v(\cdot) := u(\cdot) - u^\rho$ and set

$$\widetilde{\varphi}(\xi) := \varphi(\xi + u^\rho) - \varphi(u^\rho), \qquad \forall\, \xi \in \mathbb{R}.$$

Noting that

$$B(\widetilde{\varphi}\circ v)(n) = B\varphi(v + u^\rho)(n) - B\varphi(u^\rho) = B(\varphi\circ u)(n) - B(\rho - d_2)/\mathbf{G}(1),$$

and

$$z(n+1) - Az(n) = x(n+1) - Ax(n) - (I-A)x^\infty,$$

it follows easily from (6.32a) that

$$z(n+1) = Az(n) + B(\widetilde{\varphi}\circ v)(n). \tag{6.36}$$

Since,

$$\widetilde{\varphi}\circ v = \varphi\circ u - \varphi(u^\rho) = \varphi\circ u - \varphi(u^\infty) \in l^2(\mathbb{Z}_+,\mathbb{R}),$$

it follows from Lemma 6.1.5 applied to (6.36) that $\lim_{n\to\infty}\|z(n)\| = 0$ showing that $x(n)$ converges to x^∞ as $n \to \infty$.

Proof of Statement 2: It follows immediately from Theorem 5.1.2 and the proof of statement 1, that the conclusions of statement 1 are valid with $k^* = 1/|\lambda f_J(G)|$. □

We now consider the discrete-time system (6.1) with input non-linearity $\varphi \circ u$ in feedback interconnection with the integrator

$$\begin{aligned} z(n+1) &= z(n) - Cx(n) - D(\varphi \circ u)(n), \quad z(0) = z^0 \in \mathbb{R}, \\ u(n) &= z(n) - Cx(n) - D(\varphi \circ u)(n), \end{aligned}$$

where z denotes the integrator state. Equivalently,

$$x(n+1) = Ax(n) + B(\varphi \circ u)(n), \quad x(0) = x^0 \in X, \tag{6.37a}$$

$$z(n+1) = z(n) + k(\rho - d(n) - Cx(n) - D(\varphi \circ u)(n)), \quad z(0) = z^0 \in \mathbb{R}, \tag{6.37b}$$

$$u(n) = z(n) + k(\rho - d(n) - Cx(n) - D(\varphi \circ u)(n)), \quad u(0) = u^0 \in \mathbb{R}. \tag{6.37c}$$

Theorem 6.3.3. *Assume that system (6.1) is strongly stable, $1 \in \operatorname{res}(A)$ and $\mathbf{G}(1) > 0$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous and non-decreasing. Suppose that $d = d_1 + d_2\vartheta$ with $J_0 d_1 \in m^2(\mathbb{Z}_+, \mathbb{R})$ and $d_2 \in \mathbb{R}$. Let $\rho \in \mathbb{R}$, assume that $(\rho - d_2)/\mathbf{G}(1) \in \operatorname{im}\varphi$ and let (x, u) be a solution of (6.37). Under these conditions the following statements hold.*

1. *Assume that $\varphi - \varphi(0) \in \mathscr{S}(a)$ for some $a \in (0, \infty)$. Then there exists a constant $k^* \in (0, \infty]$ (depending on G, φ and ρ) such that for all $k \in (0, k^*)$, the limits $\lim_{n\to\infty} x(n) =: x^\infty$ and $\lim_{n\to\infty} u(n) =: u^\infty$ exist and satisfy $x^\infty = ((\rho - d_2)/\mathbf{G}(1))(I - A)^{-1}B$ and $\varphi(u^\infty) = (\rho - d_2)/\mathbf{G}(1)$, respectively,*

   $$e = \frac{1}{k}(\Delta_0 u - z_0\delta) \in l^2(\mathbb{Z}_+, \mathbb{R}) \quad \text{and} \quad \varphi \circ u - \varphi(u^\infty) \in l^2(\mathbb{Z}_+, \mathbb{R}).$$

   In particular, $\lim_{n\to\infty} e(n) = 0$. If $f_{J_0}(G) = 0$ (where $f_{J_0}(G)$ is given by (5.12)), then the above conclusions are valid with $k^* = \infty$.

2. *Assume that φ is globally Lipschitz continuous with Lipschitz constant $\lambda > 0$. Then the conclusions of statement 1 are valid with $k^* = 1/|\lambda f_{J_0}(G)|$ (where $1/0 := \infty$).*

3. *Under the assumption that $f_{J_0}(G) > 0$, the conclusions of statement 1 are valid with $k^* = \infty$.*

Proof. The proof of statements 1 and 2 of Theorem 6.3.3 follow in a similar way to the proof of Theorem 6.3.2 but in this case an application of Theorem 5.1.4 is required.

Proof of Statement 3: Statement 3 follows from statement 3 of Theorem 5.1.4 and arguments similar to those in the proof of statement 1. □

We now consider a discrete-time system with input and output non-linearities. Let $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ be static input and output non-linearities, respectively, let $\rho \in \mathbb{R}$, $\kappa : \mathbb{Z}_+ \to \mathbb{R}$ be a time-varying gain, d an external disturbance, and consider the discrete-time system (6.1), with input non-linearity $v = \varphi \circ u$ in feedback interconnection with the integrator $\Delta u = \kappa(\rho\vartheta - d - y)$ (where $y = \psi \circ w$), that is,

$$x(n+1) = Ax(n) + B(\varphi \circ u)(n), \qquad x(0) = x^0 \in X, \tag{6.38a}$$

$$u(n+1) = u(n) + \kappa(n)(\rho - d(n) - (\psi \circ (Cx + D(\varphi \circ u))(n))), \tag{6.38b}$$

$$u(0) = u^0 \in \mathbb{R}.$$

Then it is clear by iterating (6.38a) that

$$u = u^0\vartheta + J(\kappa(\rho\vartheta - d - \psi \circ (Cx + D(\varphi \circ u)))). \tag{6.39}$$

We define the tracking error $e(n) := \rho - d(n) - y(n)$ for all $n \in \mathbb{Z}_+$.

Theorem 6.3.4. *Assume that system* (6.1) *is strongly stable,* $1 \in \mathrm{res}(A)$, *and* $\mathbf{G}(1) > 0$. *Let* $\varphi : \mathbb{R} \to \mathbb{R}$ *and* $\psi : \mathbb{R} \to \mathbb{R}$ *be non-decreasing and globally Lipschitz continuous with Lipschitz constants* $\lambda_1 > 0$ *and* $\lambda_2 > 0$. *Suppose that* $d = d_1 + d_2\vartheta$ *with* $d_1 \in l^1(\mathbb{Z}_+, \mathbb{R})$, $d_2 \in \mathbb{R}$ *and* $n \mapsto \sum_{j=n}^{\infty} |d_1(j)| \in l^2(\mathbb{Z}_+, \mathbb{R})$, $\rho - d_2 \in \mathscr{R}(G, \varphi, \psi)$ *and* $\kappa : \mathbb{Z}_+ \to \mathbb{R}$ *is bounded and non-negative with*

$$\limsup_{n\to\infty} \kappa(n) < 1/|\lambda_1\lambda_2 f_{0,J}(G)|,$$

where $f_{0,J}(G)$ *is given by* (5.19). *Let* (x, u) *be the unique solution of* (6.38). *Then the following statements hold.*

1. *The limit* $(\varphi \circ u)^\infty := \lim_{n\to\infty} \varphi(u(n))$ *exists and is finite,* $\Delta(\varphi \circ u) \in l^2(\mathbb{Z}_+, \mathbb{R})$ *and* $\lim_{n\to\infty} x(n) = (I - A)^{-1}B(\varphi \circ u)^\infty$.

2. *If* $\kappa \notin l^1(\mathbb{Z}_+, \mathbb{R})$, *then* $\lim_{n\to\infty}(d(n) + (\psi \circ y)(n)) = \rho$, *or equivalently,*

$$\lim_{n\to\infty} e(n) = 0.$$

3. *If* $\rho - d_2$ *is an interior point of* $\mathscr{R}(G, \varphi, \psi)$, *then* u *is bounded.*

Proof. Let (x, u) be the unique solution of (6.38). Using (6.39), and invoking (6.2) and (6.3), a routine calculation, using arguments similar to those used to prove Lemma 6.3.1, shows that for u given by (6.38b),

$$u = u^0\vartheta + J(\kappa(\rho\vartheta - d - \psi(g + (G(\varphi \circ u))))), \tag{6.40}$$

where $g(n) := CA^n x^0$. In order to apply Theorem 5.2.2 to (6.40), we need to verify the relevant assumptions. It is clear (see Remark 6.2.2) that **G** satisfies assumption (A). By (6.5),

$$\left(\sum_{j=0}^{\infty} \|CA^j x^0\|^2 \right)^{1/2} \leq \beta \|x^0\|,$$

for some $\beta \geq 0$, hence $g \in l^2(\mathbb{Z}_+, \mathbb{R})$. Statements 2 and 3 now follow immediately from the application of Theorem 5.2.2 to (6.40). It remains to show statement 1 holds. Again applying Theorem 5.2.2 to (6.40), we immediately obtain that $(\varphi \circ u)^\infty := \lim_{n\to\infty} \varphi(u(n))$ exists and is finite and $\Delta(\varphi \circ u) \in l^2(\mathbb{Z}_+, \mathbb{R})$. It then follows from Lemma 6.1.6 that $\lim_{n\to\infty} x(n) = (I - A)^{-1} B (\varphi \circ u)^\infty$. □

Remark 6.3.5. Note that it is also possible to obtain a state-space version of Theorem 5.2.4 under similar assumptions to those in Theorem 6.3.4. ◇

6.4 Notes and references

The (non-trivial) example of a strongly stable discrete-time system which is not a power stable system (see Example 6.1.3), is new. Lemmas 6.1.5, 6.1.6 and 6.1.7, which give asymptotic properties of the state x, are new for discrete-time systems but analogous results for continuous-time systems are well-known. The results in §6.2 give state-space versions of the absolute stability theory developed in Chapter 4. We remark that the absolute stability results in Chapter 4 which assume a strict positive real condition (that is, $\varepsilon > 0$) apply to strongly stable state-space systems with $1 \in \mathrm{res}(A)$, whereas, those absolute stability results in Chapter 4 with non-strict positive real condition (that is, $\varepsilon = 0$) apply to power stable state-space systems. This mimics the continuous-time absolute stability theory developed in [9] and [10]; the absolute stability results in [9] being applied to strongly stable state-space systems with 0 in the resolvent set of the semigroup generator, and the absolute stability results in [10] being applied to exponentially stable state-space systems. The results in §6.3 form state-space versions of the low-gain integral control results in Chapter 5. The results in §6.3 significantly improve on results in the existing literature in the sense that here the underlying state-space system is assumed to be strongly stable (a larger class of systems than power-stable systems, see Example 6.1.3) whereas, the results in [17], [34], [37] and [42] assume that the underlying state-space system is power-stable. Furthermore, note that the results in [17] are obtained in a finite-dimensional setting, the results in [34] and [37] do not allow for output non- linearities, and the results in [42] do not consider any non-linearities. The absolute stability results with non-strict positive real condition, together with the results in §6.2 and Example 6.1.3, form the basis of §4 of [5].

Chapter 7

Steady-state gains and sample-hold discretisations of continuous-time infinite-dimensional linear systems

In this chapter we begin by discussing transfer functions of bounded linear shift-invariant operators on $L^2(\mathbb{R}_+, U)$. We next introduce the concepts of asymptotic steady-state gain, L^2-steady-state gain and step error in continuous time. Analogous concepts in discrete time were introduced in Chapter 3. Next we introduce the ideal sampling, generalised sampling and zero-order hold operators and state basic boundedness properties of each operator. Finally, under a mild assumption on the continuous-time system it is shown that the existence of the (continuous-time) L^2-steady-state gain implies the existence of the l^2-steady-state gain of the sample-hold discretisation. Moreover, (under a further natural assumption in the case of sample-hold discretisation with generalised sampling) we see that these two gains coincide.

7.1 Transfer functions of continuous-time operators

Recall that X denotes a Banach space, U denotes a Hilbert space and for $\alpha \in \mathbb{R}$, we have $\mathbb{C}_\alpha := \{s \in \mathbb{C} \mid \text{Re } s > \alpha\}$.

Definition. Let $f : \mathbb{C}_0 \to X$ be holomorphic. Setting

$$\|f\|_{H^2} := \left(\sup_{x>0} \int_{-\infty}^{\infty} \|f(x+iy)\|^2 \, dy \right)^{1/2}$$

and

$$\|f\|_{H^\infty} := \sup_{z \in \mathbb{C}_0} \|f(z)\|,$$

the Hardy-Lebesgue spaces $H^2(\mathbb{C}_0, X)$ and $H^\infty(\mathbb{C}_0, X)$ are defined to consist of all f for which $\|f\|_{H^2} < \infty$ and $\|f\|_{H^\infty} < \infty$, respectively.

Let $f \in H^2(\mathbb{C}_0, U)$. Then f has boundary values $f^*(iy) := \lim_{x \to 0+} f(x+iy)$ at almost all points $y \in \mathbb{R}$ (see [50], Theorem B, p. 85). Moreover, $f^* \in L^2(i\mathbb{R}, U)$ and $\|f^*\|_{L^2} = \|f\|_{H^2}$.

Theorem 7.1.1 (Paley-Wiener). *Let $f \in H^2(\mathbb{C}_0, U)$. Then, there exists $g \in L^2(\mathbb{R}_+, U)$ such that $\mathscr{L}g = f$ and $\|f\|_{H^2} = \sqrt{2\pi}\|g\|_{L^2}$.*

Let $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ be shift-invariant. As is well known (see, for example, [64], Theorem 2.3), G_c has a transfer function $\mathbf{G}_c \in H^\infty(\mathbb{C}_0, \mathscr{B}(U))$ in the sense that

$$(\mathscr{L}(G_c u))(s) = \mathbf{G}_c(s)(\mathscr{L}(u))(s), \qquad \forall\, u \in L^2(\mathbb{R}_+, U),\ s \in \mathbb{C}_0.$$

By shift-invariance, G_c is causal, and therefore G_c extends to a shift-invariant operator from $L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ into itself. We shall use the same symbol G_c to denote the original operator on $L^2(\mathbb{R}_+, U)$ and its shift-invariant extension to $L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$.

7.2 Continuous-time steady-state gains and step error

Let $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ be shift-invariant.

Definition. If there exists an operator $\Gamma_c \in \mathscr{B}(U)$ such that

$$\lim_{t \to \infty} (G_c(\vartheta_c \xi))(t) = \Gamma_c \xi\,, \qquad \forall \xi \in U\,,$$

then we say that Γ_c is the *asymptotic steady-state gain* of G_c. Moreover, if there exists an operator $\Gamma_c \in \mathscr{B}(U)$ such that

$$G_c(\vartheta_c \xi) - \vartheta_c \Gamma_c \xi \in L^2(\mathbb{R}_+, U)\,, \qquad \forall \xi \in U\,,$$

then Γ_c is said to be the *L^2-steady-state gain* of G_c. If the asymptotic steady-state gain or the L^2-steady-state gain of G_c exist, then, for $\xi \in U$, the function

$$\sigma_c^\xi := G_c(\vartheta_c \xi) - \vartheta_c \Gamma_c \xi$$

is said to be the *step error* associated with ξ.

The asymptotic steady-state gain and the L^2-steady-state gain may or may not exist. In contrast to the discrete-time case (see Chapter 3), the existence of one does not imply the existence of the other. Trivially, if they both exist, then they coincide. If Γ_c is the L^2-steady-state gain of G_c, then it is not guaranteed that $\sigma_c^\xi(t) \to 0$ as $t \to \infty$. However, since $\sigma_c^\xi \in L^2(\mathbb{R}_+, U)$, it follows from Proposition 2.1.13 that $\sigma_c^\xi(t)$ converges to 0 in measure as $t \to \infty$ in the sense that, for every $\varepsilon > 0$,

$$\lim_{T \to \infty} m\left(\{t \geq T : \|\sigma_c^\xi(t)\| \geq \varepsilon\}\right) = 0,$$

where m denotes the Lebesgue measure.

Of course, under the additonal assumption that G_c is the input-output operator of a finite-dimensional state-space system (i.e., $\mathbf{G}_c$ is rational), the asymptotic steady-state gain and the L^2-steady-state gain exist and are given by $\mathbf{G}_c(0)$; furthermore, there exist $M > 0$ and $\alpha < 0$ such that $\|\sigma_c^\xi(t)\| \leq Me^{\alpha t}\|\xi\|$ for all $t \in \mathbb{R}_+$ and for all $\xi \in U = \mathbb{R}^m$.

We introduce the following assumption on the transfer function $\mathbf{G_c}$ of G_c.

($\mathbf{A_c}$) There exists $\Gamma_c \in \mathscr{B}(U)$ such that

$$\limsup_{s \to 0, s \in \mathbb{C}_0} \left\| \frac{1}{s}(\mathbf{G}_c(s) - \Gamma_c) \right\| < \infty. \tag{7.1}$$

Remark 7.2.1. If $\mathbf{G}_c$ extends analytically into a neighbourhood of 0 (which in particular is the case if $G_c \in \mathscr{B}(L^2_\alpha(\mathbb{R}_+, U))$ for some $\alpha < 0$), then (A_c) holds with $\Gamma_c = \mathbf{G}_c(0)$. Furthermore, if $\mathbf{G}_c$ is the transfer function of a strongly stable well-posed state-space system (see Lemma 9.1.6 for more details) with the additional property that 0 is in the resolvent set of the semigroup generator (which is trivially true for exponentially stable well-posed systems), then (A_c) holds. ◇

Note that if $\mathbf{G}_c$ satisfies assumption (A_c), then it follows easily from Theorem 7.1.1 that there exists a constant $\gamma > 0$ such that

$$\|\sigma_c^\xi\|_{L^2} \leq \gamma \|\xi\|, \qquad \forall \xi \in U,$$

showing that the operator $\Sigma_c : \xi \mapsto \sigma_c^\xi$ is in $\mathscr{B}(U, L^2(\mathbb{R}_+, U))$. In particular, if assumption (A_c) holds, then Γ_c is the L^2-steady-state gain of G_c. Using Remark

7.2.1, it can be easily shown that, if $G_c \in \mathscr{B}(L^2_\alpha(\mathbb{R}_+, U))$ for some $\alpha < 0$, then $\Sigma_c \in \mathscr{B}(U, L^2_\alpha(\mathbb{R}_+, U))$.

The next result gives a time-domain characterization of assumption (A_c).

Lemma 7.2.2. *Let $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ be shift-invariant with transfer function $\mathbf{G}_c$ and let $\Gamma_c \in \mathscr{B}(U)$. Then (7.1) holds (i.e., $\mathbf{G}_c$ satisfies assumption (A_c)) if and only if $G_c\mathscr{I} - \Gamma_c\mathscr{I} \in \mathscr{B}(L^2(\mathbb{R}_+, U))$.*

Proof. $(\Rightarrow)$ Suppose that $\mathbf{G}_c$ satisfies assumption (A_c). Consider the operator $G_c\mathscr{I} - \Gamma_c\mathscr{I}$. By shift-invariance of G_c and $\mathscr{I}$ it follows that $G_c\mathscr{I} - \Gamma_c\mathscr{I}$ is shift-invariant and its transfer function is given by

$$(\mathbf{G}_c(s) - \Gamma_c)/s, \qquad s \in \mathbb{C}_0.$$

From assumption (A_c) and the fact that $\mathbf{G}_c \in H^\infty(\mathbb{C}_0, \mathscr{B}(U))$, we conclude that, $s \mapsto (\mathbf{G}_c(s) - \Gamma_c)/s \in H^\infty(\mathbb{C}_0, \mathscr{B}(U))$ and so $G_c\mathscr{I} - \Gamma_c\mathscr{I} \in \mathscr{B}(L^2(\mathbb{R}_+, U))$.
$(\Leftarrow)$ Suppose $G_c\mathscr{I} - \Gamma_c\mathscr{I} \in \mathscr{B}(L^2(\mathbb{R}_+, U))$. As before, since G_c and $\mathscr{I}$ are shift-invariant, it follows that $G_c\mathscr{I} - \Gamma_c\mathscr{I}$ is shift-invariant. Consequently, $G_c\mathscr{I} - \Gamma_c\mathscr{I}$ has a $H^\infty(\mathbb{C}_0, \mathscr{B}(U))$ transfer function. Since this transfer function is given by

$$(\mathbf{G}_c(s) - \Gamma_c)/s, \qquad s \in \mathbb{C}_0,$$

it follows that assumption (A_c) holds. □

We emphasize that assumption (A_c) does not guarantee that Γ_c is the asymptotic steady-state gain of G_c (or, equivalently, that $\sigma_c^\xi(t) \to 0$ as $t \to \infty$ for every $\xi \in U$). However, the following result holds.

Proposition 7.2.3. *Let $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ be shift-invariant with transfer function $\mathbf{G}_c$. If assumption (A_c) holds and if $\lim_{t\to\infty}(G_c u)(t) = 0$ for all $u \in C^\infty(\mathbb{R}_+, U)$ with compact support, then*

$$\lim_{t\to\infty} \sigma_c^\xi(t) = 0, \qquad \forall \xi \in U.$$

Proof. Let $\xi \in U$ be arbitrary. Choose ϑ_1 in $C^\infty(\mathbb{R}_+, \mathbb{R})$ such that

$$\vartheta_1(0) = 0, \qquad \vartheta_1(t) = 1, \qquad \forall t \geq 1.$$

Moreover, define $\vartheta_2 := \vartheta_c - \vartheta_1$. Setting $H_c := G_c\mathscr{I} - \Gamma_c\mathscr{I}$, we have that

$$G_c(\vartheta_c\xi) = G_c(\vartheta_1\xi) + G_c(\vartheta_2\xi) = H_c(\dot{\vartheta}_1\xi) + \vartheta_1\Gamma_c\xi + G_c(\vartheta_2\xi).$$

Obviously, ϑ_2 is in $C^\infty(\mathbb{R}_+, \mathbb{R})$ and has compact support. Therefore, by hypothesis, $\lim_{t\to\infty}(G_c(\vartheta_2\xi))(t) = 0$. Hence, by choice of ϑ_1,

$$\lim_{t\to\infty} (\vartheta_1(t)\Gamma_c\xi + (G_c(\vartheta_2\xi))(t)) = \Gamma_c\xi.$$

Therefore it remains to show that

$$\lim_{t\to\infty}(H_c(\dot{\vartheta}_1\xi))(t) = 0\,. \tag{7.2}$$

It follows from assumption (A_c) and Lemma 7.2.2 that $H_c \in \mathscr{B}(L^2(\mathbb{R}_+, U))$. Clearly, $\dot{\vartheta}_1 \in L^2(\mathbb{R}_+, \mathbb{R})$ and so

$$H_c(\dot{\vartheta}_1\xi) \in L^2(\mathbb{R}_+, U)\,. \tag{7.3}$$

By shift-invariance, G_c commutes with $\mathscr{I}$, and so

$$\frac{d}{dt}(H_c(\dot{\vartheta}_1\xi)) = G_c(\dot{\vartheta}_1\xi) - \dot{\vartheta}_1\Gamma_c\xi \in L^2(\mathbb{R}_+, U)\,.$$

Combining this with (7.3), we conclude from Proposition 2.1.14 that (7.2) holds. □

Furthermore, we have the following result on the behaviour of $G_c u$ for converging inputs u.

Proposition 7.2.4. *Let $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ be shift-invariant with transfer function $\mathbf{G}_c$. Assume that* (A_c) *is satisfied.*

(a) *If $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ is such that $u^\infty := \lim_{t\to\infty} u(t)$ exists and $u - \vartheta_c u^\infty \in L^2(\mathbb{R}_+, U)$, then $G_c u - \vartheta_c\Gamma_c u^\infty \in L^2(\mathbb{R}_+, U)$.*

(b) *Assume that G_c has the additional property that $\lim_{t\to\infty}(G_c u)(t) = 0$ for all $u \in C^\infty(\mathbb{R}_+, U)$ with compact support. If $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, U)$ is such that $\dot{u} \in L^2(\mathbb{R}_+, U)$ and $u^\infty := \lim_{t\to\infty} u(t)$ exists, then $\lim_{t\to\infty}(G_c u)(t) = \Gamma_c u^\infty$.*

Proof. (a) Note that,

$$G_c u - \vartheta_c\Gamma_c u^\infty = G_c(u - \vartheta_c u^\infty) + G_c\vartheta_c u^\infty - \vartheta_c\Gamma_c u^\infty. \tag{7.4}$$

Since by assumption $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ and $u - \vartheta_c u^\infty \in L^2(\mathbb{R}_+, U)$ it follows that $G_c(u - \vartheta_c u^\infty) \in L^2(\mathbb{R}_+, U)$. Furthermore, since assumption (A_c) holds, Γ_c is the L^2-steady-state gain of G_c and so $G_c\vartheta_c u^\infty - \vartheta_c\Gamma_c u^\infty \in L^2(\mathbb{R}_+, U)$. Consequently, it follows from (7.4) that $G_c u - \vartheta_c\Gamma_c u^\infty \in L^2(\mathbb{R}_+, U)$.

(b) Setting $H_c := G_c\mathscr{I} - \Gamma_c\mathscr{I}$, it follows that,

$$H_c\dot{u} = G_c u - \Gamma_c u - (G_c\vartheta_c u(0) - \Gamma_c\vartheta_c u(0)) = G_c u - \Gamma_c u - \sigma_c^{u(0)}. \tag{7.5}$$

It follows from assumption (A_c) and Lemma 7.2.2 that $H_c \in \mathscr{B}(L^2(\mathbb{R}_+, U))$. By assumption $\dot{u} \in L^2(\mathbb{R}_+, U)$ and so

$$H_c\dot{u} \in L^2(\mathbb{R}_+, U). \tag{7.6}$$

By shift-invariance G_c commutes with $\mathscr{I}$, and so

$$\frac{d}{dt}(H\dot{u}) = G_c\dot{u} - \Gamma_c\dot{u} \in L^2(\mathbb{R}_+, U). \tag{7.7}$$

Combining (7.6) and (7.7), we see from Proposition 2.1.14 that $\lim_{t\to\infty}(H\dot{u})(t) = 0$. Consequently, it follows from (7.5) that

$$\lim_{t\to\infty}((G_c u)(t) - \Gamma_c u(t) - \sigma_c^{u(0)}(t)) = 0. \tag{7.8}$$

Invoking Proposition 7.2.3 with $\xi = u(0)$ we see that $\lim_{t\to\infty} \sigma_c^{u(0)}(t) = 0$ and so, from (7.8) we have

$$\lim_{t\to\infty}(G_c u)(t) = \lim_{t\to\infty}\Gamma_c u(t) = \Gamma_c u^\infty.$$

□

7.3 Hold and sample operators

We begin this section by introducing the notion of a hold operator, an ideal sampling operator and a generalised sampling operator. Let $\tau > 0$ denote the sampling period.

Definition. The *hold operator* $\mathcal{H} : F(\mathbb{Z}_+, U) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ is given by

$$(\mathcal{H}u)(t) := u(n), \qquad t \in [n\tau, (n+1)\tau).$$

The operator $\mathcal{H}$ is also known as zero-order hold. We define the *ideal-sampling operator*, $\mathcal{S}_I : F(\mathbb{R}_+, U) \to F(\mathbb{Z}_+, U)$ by

$$(\mathcal{S}_I y)(n) := y(n\tau), \qquad \forall\, n \in \mathbb{Z}_+.$$

For a given function $w \in L^2(0, \tau)$ (said to be the weighting function) with the property

$$w(s) \geq 0, \quad \text{a.e. } s \in [0, \tau],$$

the *generalised sampling operator* $\mathcal{S} : L^2_{\mathrm{loc}}(\mathbb{R}_+, U) \to F(\mathbb{Z}_+, U)$ is defined by

$$(\mathcal{S}y)(n) := \begin{cases} 0, & \text{if } n = 0, \\ \int_0^\tau w(s) y((n-1)\tau + s)\, ds\ , & \text{if } n \geq 1. \end{cases}$$

Remark 7.3.1. It is also possible to define a generalised sampling operator

$\widetilde{S} : L^2_{\mathrm{loc}}(\mathbb{R}_+, U) \to F(\mathbb{Z}_+, U)$ by

$$(\widetilde{S}y)(n) := \int_0^\tau w(s)y(n\tau + s)\,ds, \quad \forall\, n \in \mathbb{Z}_+.$$

We note that all the results in this chapter remain true if S is replaced by $\widetilde{S}$. However, if generalised sampler $\widetilde{S}$ is used in sampled-data control (see Figure 7.1), then the output of the discrete-time controller K (a causal operator from $F(\mathbb{Z}_+, U)$ into itself) is given by the sequence $K(\widetilde{S}(y))$, where y is the continuous-time plant output. In combination with zero-order hold this leads to a continuous-time control signal u of the form

$$u = f + \mathcal{H}K(\widetilde{S}y)\,,$$

for some given $f \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$. To compute $u(t)$ for $t \in [n\tau, (n+1)\tau)$, knowledge of $(K(\widetilde{S}y))(n)$ is required at time $t = n\tau$. However, whilst $(\widetilde{S}y)(n)$ is available at time $t = (n+1)\tau$, it is not known at time $t = n\tau$. Therefore $(K(\widetilde{S}y))(n)$ should only depend on $(\widetilde{S}y)(j)$ for $0 \le j \le n-1$, but not on $(\widetilde{S}y)(n)$. We conclude that K needs to be strictly causal for the sampled-data system to be well-defined. With generalised sampler given by S, the assumption of strict causality of the discrete-time controller K is not required. $\diamond$

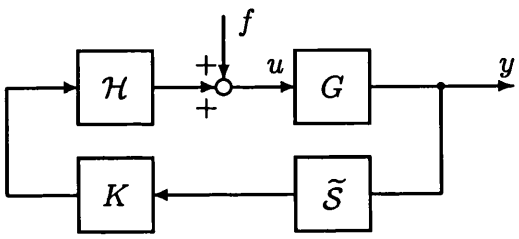

Figure 7.1: Sampled-data feedback system

Proposition 7.3.2. *Let $\alpha \in \mathbb{R}$ and set $\beta := e^{\alpha\tau}$. Then the hold and sampling operators have the following properties:*

(i) $\mathcal{H} \in \mathscr{B}(l^2_\beta(\mathbb{Z}_+, U),\ L^2_\alpha(\mathbb{R}_+, U))$;

(ii) $S \in \mathscr{B}(L^2_\alpha(\mathbb{R}_+, U),\ l^2_\beta(\mathbb{Z}_+, U))$;

(iii) $S_I \in \mathscr{B}(W^{1,2}(\mathbb{R}_+, U),\ l^2(\mathbb{Z}_+, U))$.

Proof. (i) The linearity of $\mathcal{H}$ is clear from the definition of $\mathcal{H}$. Let $\alpha \in \mathbb{R}$ and set $\beta := e^{\alpha\tau}$. Let $u \in l^2_\beta(\mathbb{Z}_+, U)$. First note that,

$$\|(\mathcal{H}u)(t)e^{-\alpha t}\|^2 = \|u(n)e^{-\alpha t}\|^2, \qquad \forall\, t \in [n\tau, (n+1)\tau).$$

Using the estimate $e^{-\alpha t} \le e^{|\alpha|\tau} e^{-\alpha n\tau}$ for all $t \in [n\tau, (n+1)\tau)$, we obtain

$$\|(\mathcal{H}u)(t)e^{-\alpha t}\|^2 \le e^{2|\alpha|\tau}\|u(n)e^{-\alpha n\tau}\|^2 = e^{2|\alpha|\tau}\|u(n)\beta^{-n}\|^2, \qquad \forall\, t \in [n\tau, (n+1)\tau).$$

Hence it follows that $\int_0^\infty \|(\mathcal{H}u)(t)e^{-\alpha t}\|^2\,dt \le \tau e^{2|\alpha|\tau}\sum_{j=0}^{\infty}\|u(j)\beta^{-j}\|^2$

$$\int_0^\infty \|(\mathcal{H}u)(t)e^{-\alpha t}\|^2\,dt \le \tau e^{2|\alpha|\tau}\sum_{j=0}^{\infty}\|u(j)\beta^{-j}\|^2$$

and so, $\|\mathcal{H}u\|_{L^2_\alpha} \le \sqrt{\tau}e^{|\alpha|\tau}\|u\|_{l^2_\beta}$ and $\mathcal{H} \in \mathscr{B}(l^2_\beta(\mathbb{Z}_+, U), L^2_\alpha(\mathbb{R}_+, U))$.

(ii) Linearity of $\mathcal{S}$ is clear by the definition of $\mathcal{S}$. Let $y \in L^2_\alpha(\mathbb{R}_+, U)$ and set $\beta := e^{\alpha\tau}$. For $n \ge 1$,

$$\begin{aligned}
\|(\mathcal{S}y)(n)\beta^{-n}\| &= \left\|\left(\int_0^\tau w(s)y((n-1)\tau+s)\,ds\right)e^{-\alpha n\tau}\right\| \\
&= \left\|\left(\int_0^\tau w(s)y((n-1)\tau+s)\,ds\right)e^{-\alpha(n-1)\tau}e^{-\alpha\tau}\right\| \\
&\le e^{|\alpha|\tau}\left\|\int_0^\tau w(s)y((n-1)\tau+s)e^{-\alpha((n-1)\tau+s)}\,ds\right\| \\
&\le e^{|\alpha|\tau}\int_0^\tau \|w(s)y((n-1)\tau+s)e^{-\alpha((n-1)\tau+s)}\|\,ds \\
&\le e^{|\alpha|\tau}\|w\|_{L^2(0,\tau)}\left(\int_0^\tau \|y((n-1)\tau+s)e^{-\alpha((n-1)\tau+s)}\|^2\,ds\right)^{1/2} \\
&= e^{|\alpha|\tau}\|w\|_{L^2(0,\tau)}\left(\int_{(n-1)\tau}^{n\tau}\|y(s)e^{-\alpha s}\|^2\,ds\right)^{1/2}
\end{aligned}$$

where, the first inequality follows from the estimate $e^{-\alpha\tau} \le e^{|\alpha|\tau}e^{-\alpha s}$, for all $s \in [0,\tau]$ and the third inequality follows from the fact that $w \in L^2(0,\tau)$, $y \in L^2_\alpha(\mathbb{R}_+, U)$ and Hölder's inequality. Hence it follows that

$$\begin{aligned}
\sum_{n=0}^{\infty}\|(\mathcal{S}y)(n)\beta^{-n}\|^2 &= \sum_{n=1}^{\infty}\|(\mathcal{S}y)(n)\beta^{-n}\|^2 \\
&\le e^{|\alpha|\tau}\|w\|^2_{L^2(0,\tau)}\sum_{n=1}^{\infty}\int_{(n-1)\tau}^{n\tau}\|y(s)e^{-\alpha s}\|^2\,ds \\
&= e^{|\alpha|\tau}\|w\|^2_{L^2(0,\tau)}\int_0^\infty \|y(s)e^{-\alpha s}\|^2\,ds.
\end{aligned}$$

Consequently,

$$\|\mathcal{S}y\|_{l^2_\beta} \le e^{\frac{|\alpha|\tau}{2}}\|w\|_{L^2(0,\tau)}\|y\|_{L^2_\alpha}$$

and we see that $\mathcal{S} \in \mathscr{B}(L^2_\alpha(\mathbb{R}_+, U), l^2_\beta(\mathbb{Z}_+, U))$.
(iii) Let $f \in W^{1,2}(\mathbb{R}_+, U)$. We define an extension g of f in the following way,

$$g(t) := \begin{cases} f(t), & t \geq 0, \\ f(-t), & t < 0. \end{cases}$$

Then g is an even function, $g \in W^{1,2}(\mathbb{R}, U)$ and $\|g\|_{W^{1,2}(\mathbb{R},U)} = 2\|f\|_{W^{1,2}(\mathbb{R}_+,U)}$. Applying Proposition 2.1 of [30] (with $s = 1$ in the notation of [30]), we deduce that

$$\|\mathcal{S}_I g\|_{l^2(\mathbb{Z},U)} \leq \gamma \|g\|_{W^{1,2}(\mathbb{R},U)}, \tag{7.9}$$

for some $\gamma > 0$. Since g is even, $\mathcal{S}_I g$ is even and $\|\mathcal{S}_I g\|_{l^2(\mathbb{Z},U)} = 2\|\mathcal{S}_I f\|_{l^2(\mathbb{Z},U)}$. Consequently, it follows from (7.9) that

$$\|\mathcal{S}_I f\|_{l^2(\mathbb{Z}_+,U)} \leq \gamma \|f\|_{W^{1,2}(\mathbb{R}_+,U)},$$

that is, $\mathcal{S}_I \in \mathscr{B}(W^{1,2}(\mathbb{R}_+, U), l^2(\mathbb{Z}_+, U))$. □

Where appropriate, we shall impose the following assumption on the shift-invariant operator $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, U))$.

(B$_c$)

$$\lim_{t\to\infty} (G_c v)(t) = 0, \quad \forall\, v \in PC(\mathbb{R}_+, U) \cap L^2(\mathbb{R}_+, U) \text{ with } v(t) \to 0 \text{ as } t \to \infty. \tag{7.10}$$

It will be explicitly stated when we assume assumption (B$_c$) holds.

Remark 7.3.3. If $U = \mathbb{R}^m$ for some $m \in \mathbb{N}$, a sufficient condition for assumption (B$_c$) to hold is that the convolution kernel of G_c is a $\mathbb{R}^{m\times m}$-valued Borel measure on $\mathbb{R}_+$ (see, for example, [21], Theorem 6.1 part (ii), p. 96). ◇

We require the following notation. Recall that $\tau > 0$ denotes the sampling period. Suppose that $t \in \mathbb{R}_+$. Then $t \in [n_t\tau, (n_t + 1)\tau)$, where

$$n_t := \lfloor t/\tau \rfloor \tag{7.11}$$

is the integer part of t/τ, that is, the greatest integer m such that $m \leq t/\tau$.

We have the following result on the behaviour of $G_c\mathcal{H}v$ for converging sequences v.

Proposition 7.3.4. *Let $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ be shift-invariant with transfer function $\mathbf{G}_c$ satisfying assumption (A$_c$). If $v \in F(\mathbb{Z}_+, U)$ is such that $\Delta v \in l^2(\mathbb{Z}_+, U)$ and $v^\infty := \lim_{n\to\infty} v(n)$ exists, then*

$$G_c\mathcal{H}v = k_{c1} + k_{c2}$$

with $\lim_{t\to\infty} k_{c1}(t) = \Gamma_c v^\infty$ *and* $k_{c2} \in L^2(\mathbb{R}_+, U)$. *If additionally* (B$_c$) *holds then,* $\lim_{t\to\infty} k_{c2}(t) = 0$ *and*

$$\lim_{t\to\infty} (G_c(\mathcal{H}v))(t) = \Gamma_c v^\infty.$$

Proof. Note that for all $t \in \mathbb{R}_+$

$$\begin{aligned}(\mathscr{I}(\mathcal{H}(\Delta v)))(t) &= \tau(J(\Delta v))(n_t) + (t - n_t\tau)(\mathcal{H}(\Delta v))(t) \\ &= \tau v(n_t - 1) - \tau\vartheta_c(t)v(0) + (t - n_t\tau)(\mathcal{H}(\Delta v))(t) \\ &= \tau(\mathcal{H}v)(t) - \tau\vartheta_c(t)v(0) + (t - n_t\tau)(\mathcal{H}(\Delta v))(t), \qquad (7.12)\end{aligned}$$

where n_t is given by (7.11). Set $h_c(t) := (t - n_t\tau)(\mathcal{H}(\Delta v))(t)$ for all $t \in \mathbb{R}_+$. Using (7.12) it follows that, for $t \in \mathbb{R}_+$,

$$(G_c(\mathscr{I}(\mathcal{H}(\Delta v))))(t) = \tau(G_c(\mathcal{H}v))(t) - \tau(G_c(\vartheta_c v(0)))(t) + (G_c h_c)(t). \qquad (7.13)$$

Setting $H_c := G_c\mathscr{I} - \Gamma_c\mathscr{I}$, it follows from (7.12) and (7.13), with rearrangement, that

$$\begin{aligned}G_c(\mathcal{H}v) = {}& \frac{1}{\tau}H_c(\mathcal{H}(\Delta v)) + \frac{1}{\tau}\Gamma_c h_c + \Gamma_c\mathcal{H}v \\ &+ G_c(\vartheta_c v(0)) - \Gamma_c v(0)\vartheta_c - \frac{1}{\tau}G_c h_c. \qquad (7.14)\end{aligned}$$

With

$$k_{c1} := \frac{1}{\tau}H_c(\mathcal{H}(\Delta v)) + \frac{1}{\tau}\Gamma_c h_c + \Gamma_c\mathcal{H}v$$

and

$$k_{c2} := G_c(\vartheta_c v(0)) - \Gamma_c v(0)\vartheta_c - \frac{1}{\tau}G_c h_c,$$

it is clear from (7.14) that $G_c(\mathcal{H}v) = k_{c1} + k_{c2}$. We claim that

$$\lim_{t\to\infty} k_{c1}(t) = \Gamma_c v^\infty. \qquad (7.15)$$

To this end note that since $\Delta v \in l^2(\mathbb{Z}_+, U)$, it follows that $\mathcal{H}(\Delta v) \in L^2(\mathbb{R}_+, U)$ and $(\mathcal{H}(\Delta v))(t) \to 0$ as $t \to \infty$. Noting that the function

$$t \mapsto t - n_t\tau = t - \tau\lfloor t/\tau \rfloor, \quad t \in \mathbb{R}_+$$

is bounded (it takes values in $[0, \tau)$), it follows that $h_c \in L^2(\mathbb{R}_+, U)$ and moreover,

$$\lim_{t\to\infty} h_c(t) = 0. \qquad (7.16)$$

It follows from assumption (A$_c$) and Lemma 7.2.2 that $H_c \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ and so $H_c(\mathcal{H}(\Delta v)) \in L^2(\mathbb{R}_+, U)$. Furthermore, the derivative of $H_c(\mathcal{H}(\Delta v))$, namely

$G_c\mathcal{H}\triangle v - \Gamma_c\mathcal{H}\triangle v$, is in $L^2(\mathbb{R}_+, U)$ and we deduce that

$$\lim_{t\to\infty}(H_c(\mathcal{H}(\triangle v)))(t) = 0. \tag{7.17}$$

Combining (7.16), (7.17) with the fact that $v^\infty := \lim_{n\to\infty} v(n)$ exists, it follows that (7.15) holds. It remains to show that $k_{c2} \in L^2(\mathbb{R}_+, U)$. Since $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ and $h_c \in L^2(\mathbb{R}_+, U)$ it is clear that $G_c h_c \in L^2(\mathbb{R}_+, U)$. Furthermore,

$$G_c(\vartheta_c v(0)) - \Gamma_c v(0)\vartheta_c = \sigma_c^{v(0)}$$

and, by assumption (A_c), $\sigma_c^{v(0)} \in L^2(\mathbb{R}_+, U)$. Consequently, $k_{c2} \in L^2(\mathbb{R}_+, U)$. If additionally ($\mathrm{B}_c$) holds, replacing ξ by $v(0)$ in Proposition 7.2.3, it follows that

$$\lim_{t\to\infty} \sigma_c^{v(0)}(t) = \lim_{t\to\infty}[(G_c(\vartheta_c v(0)))(t) - \Gamma_c v(0)\vartheta_c(t)] = 0. \tag{7.18}$$

Using (7.16) and the fact that $h_c \in PC(\mathbb{R}_+, U) \cap L^2(\mathbb{R}_+, U)$, we obtain

$$\lim_{t\to\infty}(G_c h_c)(t) = 0.$$

Consequently, it is clear that given (B_c) holds, $\lim_{t\to\infty} k_{c2}(t) = 0$ and

$$\lim_{t\to\infty}(G_c(\mathcal{H}v))(t) = \lim_{t\to\infty} k_{c1}(t) = \Gamma_c v^\infty.$$

□

7.4 Sample-hold discretisations

Sample-hold discretisation with generalised sampling

Let $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ be shift-invariant and let $u \in L^2(\mathbb{R}_+, U)$. Due to the potential irregularity of $G_c u$, ideal sampling of $G_c u$ is meaningless. Therefore we consider generalised sampling of $G_c u$. The sample-hold discretisation G of G_c is defined by

$$G := \mathcal{S}G_c\mathcal{H}. \tag{7.19}$$

Proposition 7.4.1. *Let $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ be shift-invariant. Then for G given by* (7.19)*, $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ and is shift-invariant.*

Proof. Boundedness and linearity of G follow immediately from Proposition 7.3.2. Let $u \in l^2(\mathbb{Z}_+, U)$. To see that G is shift-invariant we must show that

$$(\mathbf{S}Gu)(n) = (G\mathbf{S}u)(n), \qquad \forall\, n \in \mathbb{Z}_+.$$

We consider the following cases.
CASE 1: $n = 0$. It is clear by the definition of $\mathbf{S}$ and G that

$$0 = ((\mathbf{S}G)u)(0) = ((G\mathbf{S})u)(0).$$

CASE 2: $n = 1$. First note that

$$(\mathbf{S}(Gu))(1) = (Gu)(0) = (\mathcal{S}(G_c\mathcal{H}u))(0) = 0.$$

By definition of G,

$$(G(\mathbf{S}u))(1) = \int_0^\tau w(s)(G_c\mathcal{H}\mathbf{S}u)(s)\, ds.$$

Noting that

$$(\mathcal{H}\mathbf{S}u)(s) = (\mathbf{S}u)(0) = 0, \qquad \forall\, s \in [0, \tau),$$

it follows from causality of G_c that $(G_c\mathcal{H}\mathbf{S}u)(s) = 0$ for all $s \in [0, \tau)$. Hence we see that $(\mathbf{S}Gu)(1) = (G\mathbf{S}u)(1) = 0$.
CASE 3: $n \geq 2$. We note that

$$\begin{aligned}(\mathbf{S}(Gu))(n) = (Gu)(n-1) &= \int_0^\tau w(s)(G_c\mathcal{H}u)((n-2)\tau + s)\, ds \\ &= \int_0^\tau w(s)(\mathbf{S}_\tau G_c\mathcal{H}u)((n-1)\tau + s)\, ds.\end{aligned}$$

Consequently, by shift-invariance of G_c it follows that

$$(\mathbf{S}(Gu))(n) = \int_0^\tau w(s)(G_c\mathbf{S}_\tau\mathcal{H}u)((n-1)\tau + s)\, ds.$$

Noting that

$$(G(\mathbf{S}u))(n) = \int_0^\tau w(s)(G_c\mathcal{H}\mathbf{S}u)((n-1)\tau + s)\, ds,$$

it remains to show that $G_c\mathbf{S}_\tau\mathcal{H} = G_c\mathcal{H}\mathbf{S}$. Let $t \geq \tau$. Then $t \in [n_t\tau, (n_t + 1)\tau)$, where $n_t \geq 1$ is given by (7.11). By definition of $\mathcal{H}$, $\mathbf{S}$ and $\mathbf{S}_\tau$,

$$(\mathcal{H}\mathbf{S}u)(t) = (\mathbf{S}u)(n_t) = u(n_t - 1),$$

and

$$(\mathbf{S}_\tau\mathcal{H}u)(t) = (\mathcal{H}u)(t - \tau) = u(n_t - 1).$$

So we have that

$$(\mathbf{S}_\tau\mathcal{H}u)(t) = (\mathcal{H}\mathbf{S}u)(t), \qquad \forall\, t \geq \tau.$$

Furthermore, if $t \in [0, \tau)$, then

$$(\mathbf{S}_\tau \mathcal{H}u)(t) = 0 = (\mathcal{H}\mathbf{S}u)(t).$$

Consequently,

$$(\mathbf{S}_\tau \mathcal{H}u)(t) = (\mathcal{H}\mathbf{S}u)(t), \qquad \forall\, t \in \mathbb{R}_+$$

and so we see that

$$(G_c(\mathbf{S}_\tau \mathcal{H}u))(t) = (G_c(\mathcal{H}\mathbf{S}u))(t), \qquad \forall\, t \in \mathbb{R}_+.$$

Hence $((\mathbf{S}G)u)(n) = ((G\mathbf{S})u)(n)$ for all $u \in l^2(\mathbb{Z}_+, U)$, $n \in \mathbb{Z}_+$ and we see that G is shift-invariant. □

We require some preliminary results.

Lemma 7.4.2. $\mathscr{I}\mathcal{H} - \tau\mathcal{H}J_0 \in \mathscr{B}(l^2(\mathbb{Z}_+, U), L^2(\mathbb{R}_+, U))$.

Proof. Let $t \in \mathbb{R}_+$ and $u \in l^2(\mathbb{Z}_+, U)$. Then by definition of $\mathcal{H}$,

$$(\mathscr{I}(\mathcal{H}u))(t) = \int_0^{n_t\tau} (\mathcal{H}u)(s)\,ds + \int_{n_t\tau}^t (\mathcal{H}u)(s)\,ds = \tau(Ju)(n_t) + (t - n_t\tau)u(n_t)$$

and

$$(\mathcal{H}(J_0u))(t) = (J_0u)(n_t),$$

where n_t is given by (7.11). Hence it follows that

$$\begin{aligned}
\|((\mathscr{I}\mathcal{H} - \tau\mathcal{H}J_0)u)(t)\| &= \|\tau(Ju)(n_t) + (t - n_t\tau)u(n_t) - \tau(J_0u)(n_t)\| \\
&= \|(t - n_t\tau)u(n_t) - \tau u(n_t)\| \\
&\le (t - n_t\tau)\|u(n_t)\| + \tau\|u(n_t)\| \\
&\le 2\tau\|u(n_t)\|, \qquad \forall\, t \in \mathbb{R}_+.
\end{aligned} \tag{7.20}$$

By (7.20)

$$\int_0^\infty \|((\mathscr{I}\mathcal{H} - \tau\mathcal{H}J_0)u)(t)\|^2\,dt \le 4\tau^3 \sum_{n=0}^\infty \|u(n)\|^2.$$

Hence it follows that

$$\|(\mathscr{I}\mathcal{H} - \tau\mathcal{H}J_0)u\|_{L^2} \le 2\tau^{3/2}\|u\|_{l^2}$$

and we see that $\mathscr{I}\mathcal{H} - \tau\mathcal{H}J_0 \in \mathscr{B}(l^2(\mathbb{Z}_+, U),\, L^2(\mathbb{R}_+, U))$. □

Lemma 7.4.3. *Let $w \in L^2(0, \tau)$. Then*

$$\left(\int_0^\tau w(s)\,ds\right)\mathbf{S} = \mathcal{S}\mathcal{H}.$$

Proof. Let $u \in l^2(\mathbb{Z}_+, U)$. We shall show that

$$\left(\int_0^\tau w(s)\, ds\right)(\mathbf{S}u)(n) = (\mathcal{S}\mathcal{H}u)(n) \qquad \forall\, n \in \mathbb{Z}_+.$$

CASE 1: $n = 0$.
Clearly $(\mathcal{S}(\mathcal{H}u))(0) = 0$ by the definition of $\mathcal{S}$ and also $(\int_0^\tau w(s)\, ds)(\mathbf{S}u)(0) = 0$ by definition of $\mathbf{S}$.
CASE 2: $n \geq 1$.
We have

$$(\mathcal{S}(\mathcal{H}u))(n) = \int_0^\tau w(s)(\mathcal{H}u)((n-1)\tau + s)\, ds. \tag{7.21}$$

Consider $(\mathcal{H}u)((n-1)\tau + s)$ where $0 \leq s < \tau$. Set $t = (n-1)\tau + s$. Then $t \in [(n-1)\tau, n\tau)$ and by definition of $\mathcal{H}$,

$$(\mathcal{H}u)(t) = u(n-1).$$

Hence from (7.21) we see that

$$\begin{aligned}(\mathcal{S}(\mathcal{H}u))(n) = \int_0^\tau w(s)u(n-1)\, ds &= \left(\int_0^\tau w(s)\, ds\right)u(n-1)\\ &= \left(\int_0^\tau w(s)\, ds\right)(\mathbf{S}u)(n)\end{aligned}$$

which shows $(\int_0^\tau w(s)\, ds)\mathbf{S} = \mathcal{S}\mathcal{H}$. □

The following result is the key result of this subsection and shows that if assumption (A$_c$) holds for G_c, then G given by (7.19) satisfies assumption (A).

Theorem 7.4.4. *Let $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, U))$ be shift-invariant and suppose that $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ is given by* (7.19). *If the transfer function $\mathbf{G}_c$ of G_c satisfies assumption* (A$_c$), *then the transfer function $\mathbf{G}$ of G satisfies assumption* (A) *with $\Gamma = (\int_0^\tau w(s)\, ds)\Gamma_c$.*

Remark 7.4.5. If additionally

$$\int_0^\tau w(s)\, ds = 1,$$

it follows from Theorem 7.4.4 that the L^2-steady-state gain of G_c and l^2-steady-state gain of G coincide. ◇

Proof of Theorem 7.4.4. By Lemma 3.4.2 it is sufficient to show that

$$GJ_0 - \Gamma J_0 \in \mathscr{B}(l^2(\mathbb{Z}_+, U)). \tag{7.22}$$

Since (A_c) holds, it follows from Lemma 7.2.2 and Proposition 7.3.2 that

$$\mathcal{S}(G_c\mathscr{I} - \Gamma_c\mathscr{I})\mathcal{H} \in \mathscr{B}(l^2(\mathbb{Z}_+, U)).$$

Therefore, (7.22) is equivalent to the claim

$$GJ_0 - \Gamma J_0 - \frac{1}{\tau}\mathcal{S}(G_c\mathscr{I} - \Gamma_c\mathscr{I})\mathcal{H} \in \mathscr{B}(l^2(\mathbb{Z}_+, U)). \tag{7.23}$$

Using (7.19) we have,

$$\begin{aligned} GJ_0 - \Gamma J_0 &- \frac{1}{\tau}\mathcal{S}(G_c\mathscr{I} - \Gamma_c\mathscr{I})\mathcal{H} \\ &= \mathcal{S}G_c\mathcal{H}J_0 - \Gamma J_0 - \frac{1}{\tau}\mathcal{S}G_c\mathscr{I}\mathcal{H} + \frac{1}{\tau}\Gamma_c\mathcal{S}\mathscr{I}\mathcal{H} \\ &= \frac{1}{\tau}\mathcal{S}G_c(\tau\mathcal{H}J_0 - \mathscr{I}\mathcal{H}) - \Gamma J_0 + \frac{1}{\tau}\Gamma_c\mathcal{S}\mathscr{I}\mathcal{H}. \end{aligned} \tag{7.24}$$

By Lemma 7.4.2, Proposition 7.3.2 and the fact that $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, U))$, we have

$$\frac{1}{\tau}\mathcal{S}G_c(\tau\mathcal{H}J_0 - \mathscr{I}\mathcal{H}) \in \mathscr{B}(l^2(\mathbb{Z}_+, U)).$$

Combining this with (7.24), we see that, in order to verify (7.23) (and hence (7.22)), it is sufficient to prove that

$$-\Gamma J_0 + \frac{1}{\tau}\Gamma_c\mathcal{S}\mathscr{I}\mathcal{H} \in \mathscr{B}(l^2(\mathbb{Z}_+, U)). \tag{7.25}$$

To this end note that by Lemma 7.4.3 and the definition of Γ it follows that,

$$\begin{aligned} -\Gamma J_0 + \frac{1}{\tau}\Gamma_c\mathcal{S}\mathscr{I}\mathcal{H} &= \frac{\Gamma_c}{\tau}(\mathcal{S}(\mathscr{I}\mathcal{H} - \tau\mathcal{H}J_0)) - \Gamma J_0 + \Gamma_c\mathcal{S}\mathcal{H}J_0 \\ &= \frac{\Gamma_c}{\tau}(\mathcal{S}(\mathscr{I}\mathcal{H} - \tau\mathcal{H}J_0)) - \Gamma J_0 + \left(\int_0^\tau w(s)\,ds\right)\Gamma_c\mathrm{S}J_0 \\ &= \frac{\Gamma_c}{\tau}(\mathcal{S}(\mathscr{I}\mathcal{H} - \tau\mathcal{H}J_0)) - \Gamma I. \end{aligned} \tag{7.26}$$

Consequently, by Lemma 7.4.2 and Proposition 7.3.2, it follows from (7.26) that (7.25) holds. □

Sample-hold discretisation with ideal sampling

We assume throughout this subsection that $U = \mathbb{R}^m$ for some $m \in \mathbb{N}$ and the operator G_c is given by convolution with a $\mathbb{R}^{m\times m}$-valued Borel measure μ on $\mathbb{R}_+$, that is,

$$G_c u = \mu * u, \qquad \forall\, u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m). \tag{7.27}$$

We note that by standard properties of convolution of a measure and an L^p function (see, for example, [20], p. 271), $G_c \in \mathscr{B}(L^p(\mathbb{R}_+, \mathbb{R}^m))$ for $1 \le p \le \infty$. The transfer function $\mathbf{G}_c \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m\times m})$ of G_c is given by the Laplace transform of μ, that is

$$\mathbf{G}_c(s) = (\mathscr{L}(\mu))(s) = \int_0^\infty e^{-st}\mu(dt), \qquad \forall\, s \in \mathbb{C}_0.$$

In fact, $\mathscr{L}(\mu)$ is defined and continuous on $\overline{\mathbb{C}}_0$. Therefore $\mathbf{G}_c$ extends to a continuous function on $\overline{\mathbb{C}}_0$. It is clear that G_c has an asymptotic steady-state gain $\Gamma_c \in \mathbb{R}^{m\times m}$ which is given by

$$\Gamma_c = \mathbf{G}_c(0) = (\mathscr{L}(\mu)(0)) = \mu(\mathbb{R}_+).$$

Remark 7.4.6. The specific example of the measure μ given by

$$\mu(dt) = \frac{dt}{(1+t)^{1+\varepsilon}}, \qquad \varepsilon \in (0, 1/2),$$

shows that G_c given by (7.27) does in general not have a L^2-steady-state gain. To see this we observe the following:

$$\begin{aligned}(\mu * \vartheta_c)(t) - \Gamma_c\vartheta_c(t) &= \int_0^t \frac{1}{(1+s)^{1+\varepsilon}}\,ds - \Gamma_c\vartheta_c(t)\\ &= -\frac{1}{\varepsilon}\frac{1}{(1+t)^\varepsilon} + \frac{1}{\varepsilon} - \Gamma_c.\end{aligned}$$

Since $\varepsilon \in (0, 1/2)$, it follows that

$$t \mapsto \frac{1}{(1+t)^\varepsilon} \notin L^2(\mathbb{R}_+, \mathbb{R}^m).$$

Consequently,

$$\mu * \vartheta_c - \Gamma_c\vartheta_c \notin L^2(\mathbb{R}_+, \mathbb{R}^m)$$

for any choice of Γ_c. ◇

The sample-hold discretisation G of G_c is now defined by

$$G := \mathcal{S}_I G_c \mathcal{H}. \tag{7.28}$$

Let $\{E_k\}_{k\in\mathbb{Z}_+}$ be the family of subsets of $\mathbb{R}_+$ given by

$$E_0 := \{0\}, \qquad E_k := ((k-1)\tau, k\tau], \quad k \in \mathbb{N}.$$

The following result shows that G is shift-invariant and $G \in \mathscr{B}(l^p(\mathbb{Z}_+, \mathbb{R}^m))$, for $1 \le p \le \infty$.

Proposition 7.4.7. *Let G_c be given by* (7.27)*, where μ is a $\mathbb{R}^{m\times m}$-valued Borel measure on $\mathbb{R}_+$. Then the sequence $g \in F(\mathbb{Z}_+, \mathbb{R}^{m\times m})$ given by*

$$g(k) := \mu(E_k), \qquad k \in \mathbb{Z}_+,$$

is in $l^1(\mathbb{Z}_+, \mathbb{R}^{m\times m})$, and moreover, the operator G defined by (7.28) *satisfies*

$$Gu = g * u\,, \quad \forall\, u \in F(\mathbb{Z}_+, \mathbb{R}^m).$$

Consequently, G is shift-invariant and $G \in \mathscr{B}(l^p(\mathbb{Z}_+, \mathbb{R}^m))$ for $1 \le p \le \infty$.

Proof. By direct substitution we have

$$\begin{aligned}(Gu)(k) = ((\mathcal{S}_I G_c \mathcal{H})u)(k) &= (G_c \mathcal{H}u)(k\tau)\\ &= (\mu * \mathcal{H}u)(k\tau)\\ &= \int_0^{k\tau} \mu(ds)(\mathcal{H}u)(k\tau - s)\\ &= \sum_{j=0}^{k} \int_{E_j} \mu(ds)u(k-j)\\ &= \sum_{j=0}^{k} g(k)u(k-j).\end{aligned}$$

To see that $g \in l^1(\mathbb{Z}_+, \mathbb{R}^{m\times m})$, observe that by the finiteness of $|\mu|$

$$\sum_{k=0}^{\infty} \|g(k)\| = \sum_{k=0}^{\infty} \|\mu(E_k)\| \le \sum_{k=0}^{\infty} |\mu|(E_k) = \int_0^{\infty} |\mu|(ds) \le |\mu|(\mathbb{R}_+) < \infty.$$

Hence with $u \in l^p(\mathbb{Z}_+, \mathbb{R}^m)$ for $1 \le p \le \infty$, it follows that,

$$\|Gu\|_{l^p} = \|g * u\|_{l^p} \le \|g\|_{l^1} \|u\|_{l^p}$$

and we see that $G \in \mathscr{B}(l^p(\mathbb{Z}_+, \mathbb{R}^m))$. □

Note that by Proposition 7.4.7 the convolution kernel of G is summable, that is $Gu = g * u$, where $g \in l^1(\mathbb{Z}_+, \mathbb{R}^{m\times m})$. Consequently, the transfer function

$\mathbf{G} \in H^{\infty}(\mathbb{E}_1, \mathbb{C}^{m\times m})$ is given by

$$\mathbf{G}(z) = (\mathscr{Z}(g))(z) = \sum_{n=0}^{\infty} g(n) z^{-n}, \qquad \forall\, z \in \mathbb{E}_1.$$

In fact $\mathscr{Z}(g)$ is defined on and continuous on $\overline{\mathbb{E}}_1$. Therefore, $\mathbf{G}$ extends to a continuous function on $\overline{\mathbb{E}}_1$. It is clear that G has an asymptotic steady-state gain Γ which is given by

$$\Gamma = \mathbf{G}(1) = (\mathscr{Z}(g))(1) = \sum_{n=0}^{\infty} g(n).$$

Similar to the continuous-time case (see Remark 7.4.6), G does in general not have a l^2-steady-state gain.

We require the following preliminary result.

Lemma 7.4.8. $(1/\tau)\mathcal{S}_I \mathscr{I} \mathcal{H} = J$.

Proof. Let $u \in F(\mathbb{Z}_+, \mathbb{R}^m)$. Then,

$$(\mathcal{S}_I(\mathscr{I}(\mathcal{H}u)))(n) = (\mathscr{I}(\mathcal{H}u))(n\tau) = \int_0^{n\tau} (\mathcal{H}u)(s)\, ds.$$

It follows from the $(\mathcal{S}_I(\mathscr{I}(\mathcal{H}u)))(n) = (\mathscr{I}(\mathcal{H}u))(n\tau) = \int_0^{n\tau} (\mathcal{H}u)(s)\, ds.$

$$\int_0^{n\tau} (\mathcal{H}u)(s)\, ds = \begin{cases} 0, & \text{if } n = 0, \\ \tau \sum_{j=0}^{n-1} u(j), & \text{if } n \geq 1, \end{cases}$$
$$= \tau (Ju)(n).$$

It is then clear that $(1/\tau)\mathcal{S}_I \mathscr{I} \mathcal{H} = J$. □

The following result is the ideal sampling counterpart of Theorem 7.4.4 and shows that if assumption (A_c) holds for G_c given by (7.27), then G given by (7.28) satisfies assumption (A).

Theorem 7.4.9. *Let G_c be given by (7.27), where μ is a $\mathbb{R}^{m\times m}$-valued Borel measure on $\mathbb{R}_+$, and let G be given by (7.28). If the transfer function $\mathbf{G}_c$ of G_c satisfies assumption (A_c), then the transfer function $\mathbf{G}$ of G satisfies assumption* (A) *and* $\Gamma = \mathbf{G}(1) = \mathbf{G}_c(0) = \Gamma_c$.

Remark 7.4.10. It follows from Theorem 7.4.9 that the L^2-steady-state gain of G_c and l^2-steady-state gain of G coincide. ◇

Proof of Theorem 7.4.9. By Lemma 3.4.2 it is sufficient to show that

$$G J_0 - \Gamma_c J_0 \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}^m)). \tag{7.29}$$

We first show that

$$\mathcal{S}_I(G_c\mathscr{I} - \Gamma_c\mathscr{I})\mathcal{H} \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}^m)). \tag{7.30}$$

For the remainder of this proof let $u \in l^2(\mathbb{Z}_+, \mathbb{R}^m)$. Define $f := (G_c\mathscr{I} - \Gamma_c\mathscr{I})\mathcal{H}u$. Since (A_c) holds, it follows from Lemma 7.2.2 and Proposition 7.3.2 (i) that $f \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. Furthermore, $f(0) = 0$ and, using the fact that by shift-invariance G_c and $\mathscr{I}$ commute,

$$\begin{aligned} f(t) - f(0) &= \int_0^t (G_c\mathcal{H}u)(s)\,ds - \Gamma_c \int_0^t (\mathcal{H}u)(s)\,ds \\ &= \int_0^t (((G_c - \Gamma_c I)\mathcal{H})u)(s)\,ds. \end{aligned}$$

Since $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, \mathbb{R}^m))$, invoking Proposition 7.3.2 (i), we have $(G_c - \Gamma_c I)\mathcal{H}u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. Thus $f \in W^{1,2}(\mathbb{R}_+, \mathbb{R}^m)$ and so, by Proposition 7.3.2 (iii), $\mathcal{S}_I f \in l^2(\mathbb{Z}_+, \mathbb{R}^m)$. Again since (A_c) holds, it follows from Lemma 7.2.2, the fact that $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, \mathbb{R}^m))$ and Proposition 7.3.2 (i) that

$$\begin{aligned} \|\mathcal{S}_I(G_c\mathscr{I} - \Gamma_c\mathscr{I})\mathcal{H}u\|_{l^2} &= \|\mathcal{S}_I f\|_{l^2} \\ &\leq \|\mathcal{S}_I\|\|f\|_{W^{1,2}} \\ &= \|\mathcal{S}_I\|(\|(G_c\mathscr{I} - \Gamma_c\mathscr{I})\mathcal{H}u\|_{L^2} + \|G_c\mathcal{H}u - \Gamma_c\mathcal{H}u\|_{L^2}) \\ &\leq \alpha\|u\|_{l^2} \end{aligned}$$

where $\alpha := \|\mathcal{S}_I\|\|\mathcal{H}\|(\|G_c\mathscr{I} - \Gamma_c\mathscr{I}\| + \|G_c\| + \|\Gamma_c\|)$. Hence we see that (7.30) holds. Therefore, it follows from (7.30) that (7.29) is equivalent to the claim

$$GJ_0 - \Gamma_c J_0 - \frac{1}{\tau}\mathcal{S}_I(G_c\mathscr{I} - \Gamma_c\mathscr{I})\mathcal{H} \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}^m)). \tag{7.31}$$

Using (7.28) we have,

$$GJ_0 - \Gamma_c J_0 - \frac{1}{\tau}\mathcal{S}_I(G_c\mathscr{I} - \Gamma_c\mathscr{I})\mathcal{H} = \frac{1}{\tau}\mathcal{S}_I G_c(\tau\mathcal{H}J_0 - \mathscr{I}\mathcal{H}) - \Gamma_c J_0 + \frac{1}{\tau}\Gamma_c\mathcal{S}_I\mathscr{I}\mathcal{H}. \tag{7.32}$$

Applying Lemma 7.4.8 to (7.32) and simplifying we obtain

$$GJ_0 - \Gamma_c J_0 - \frac{1}{\tau}\mathcal{S}_I(G_c\mathscr{I} - \Gamma_c\mathscr{I})\mathcal{H} = \frac{1}{\tau}\mathcal{S}_I G_c(\tau\mathcal{H}J_0 - \mathscr{I}\mathcal{H}) - \Gamma_c I. \tag{7.33}$$

Hence from (7.33) we see that in order to verify (7.31) (and hence (7.29)) it is sufficient to prove that

$$\mathcal{S}_I G_c(\tau\mathcal{H}J_0 - \mathscr{I}\mathcal{H}) \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}^m)). \tag{7.34}$$

To this end note that for all $n \in \mathbb{Z}_+$,

$$\begin{aligned}(\mathcal{S}_I(G_c(\tau\mathcal{H}J_0 - \mathscr{I}\mathcal{H}))u)(n) &= \int_0^{n\tau} \mu(ds)((\tau\mathcal{H}J_0 - \mathscr{I}\mathcal{H})u)(n\tau - s)\\ &= \sum_{k=0}^{n} \int_{E_k} \mu(ds)((\tau\mathcal{H}J_0 - \mathscr{I}\mathcal{H})u)(n\tau - s).\end{aligned}$$

It follows from the definition of E_k that, for $s \in E_k$, $n\tau - s \in [(n-k)\tau, (n-k+1)\tau)$. Consequently,

$$\begin{aligned}((\tau\mathcal{H}J_0 - \mathscr{I}\mathcal{H})u)(n\tau - s) &= \tau(J_0u)(n-k) - \tau(Ju)(n-k) + (s - k\tau)u(n-k)\\ &= (s - (k-1)\tau)u(n-k).\end{aligned}$$

Hence defining $f_k : E_k \to \mathbb{R}$ by $s \mapsto s - (k-1)\tau$, it follows that

$$(\mathcal{S}_I(G_c(\tau\mathcal{H}J_0 - \mathscr{I}\mathcal{H}))u)(n) = \sum_{k=0}^{n} \int_{E_k} \mu(ds) f_k(s) u(n-k)$$

Since $0 \le f_k(s) \le \tau$ for all $k \in \mathbb{Z}_+$, we have

$$\begin{aligned}\|(\mathcal{S}_I(G_c(\tau\mathcal{H}J_0 - \mathscr{I}\mathcal{H}))u)(n)\| &\le \sum_{k=0}^{n} \left\| \int_{E_k} \mu(ds) f_k(s) u(n-k) \right\|\\ &\le \tau \sum_{k=0}^{n} \int_{E_k} |\mu|(ds) \|u(n-k)\|\\ &= \tau \sum_{k=0}^{n} |\mu|(E_k) \|u(n-k)\|\\ &= \tau \sum_{k=0}^{n} \eta(k) \|u(n-k)\|\\ &= \tau(\eta * v)(n), \end{aligned} \tag{7.35}$$

where $\eta(k) := |\mu|(E_k)$ and $v \in F(\mathbb{Z}_+, \mathbb{R})$ is defined by $v(n) := \|u(n)\|$ for all $n \in \mathbb{Z}_+$. Noting that $\sum_{k=0}^{\infty} \|\eta(k)\| = |\mu|(\mathbb{R}_+) < \infty$ and so $\eta \in l^1(\mathbb{R}_+, \mathbb{R}^{m\times m})$, we obtain from (7.35),

$$\|(\mathcal{S}_I(G_c(\tau\mathcal{H}J_0 - \mathscr{I}\mathcal{H})))u\|_{l^2} \le \tau\|\eta * v\|_{l^2} \le \tau\|\eta\|_{l^1}\|u\|_{l^2}$$

showing that (7.34) holds. □

7.5 Notes and references

The results in this chapter are similar in spirit to [24, 25], where sample-hold discretisations of distributed parameter systems belonging to the Callier-Desoer algebra were studied using an input-output approach. In particular, Theorems 7.4.4 and 7.4.9 provide considerably more general results than those in [24, 25].

In Theorem 7.4.4 we consider the large class of continuous-time input-output operators from the algebra of shift-invariant bounded linear operators on $L^2(\mathbb{R}_+)$, together with a discretisation formed by generalised sampling combined with zero-order hold. Generalised sampling is not considered in [24, 25]. In Theorem 7.4.9 we consider continuous-time systems given by convolution with matrix-valued Borel measures defined on $\mathbb{R}_+$. This general class of measure kernels contains measures which may have singular part, a situation not considered in [24, 25]. We note that Proposition 7.3.4, Theorem 7.4.4 and Theorem 7.4.9 are particularly important in low-gain sampled-data integral control of linear infinite-dimensional systems subject to actuator and sensor non-linearities (see Chapter 8).

Statements (a) and (b) of Proposition 7.2.4, whilst not directly relevant to this thesis, are of importance in terms of analysing the behaviour of continuous-time systems subject to converging inputs.

The results in this chapter form the basis of [8] and some of [6].

Chapter 8

Sampled-data low-gain integral control in the presence of input/output non-linearities

In this chapter we apply the results from Chapter 5 to obtain sampled-data low-gain integral control results in the single-input-single-output setting. In the presence of input non-linearities we form the sample-hold discretisation of the continuous-time system with generalised sampling and zero-order hold. We then derive results on low-gain integral control for both constant and time-varying gain. In the presence of input and output non-linearities we restrict to continuous-time systems whose convolution kernel is a finite Borel measure on $\mathbb{R}_+$ and form the sample-hold discretisation of the continuous-time system with idealised sampling and zero-order hold. We then derive results on low-gain integral control with time-varying gain. The main results in this chapter are also restated for the J_0 integrator.

8.1 Sampled-data low-gain integral control in the presence of input non-linearities

In the presence of input non-linearities we distinguish two cases: constant gain and time-varying gain.

Constant gain

Consider the feedback system shown in Figure 8.1, where k is a gain parameter, the operator $G_c \in \mathscr{B}(L^2_\alpha(\mathbb{R}_+, \mathbb{R}))$ for some $\alpha < 0$, is shift-invariant with transfer function denoted by $\mathbf{G}_c$, $\varphi : \mathbb{R} \to \mathbb{R}$ is a static input non-linearity, $\rho \in \mathbb{R}$ is a constant reference value, $u^0 \in \mathbb{R}$ is the initial state of the integrator (or, equivalently, the initial value of u), the function g_c models the effect of non-zero initial conditions of the system with input-output operator G_c, the operators $\mathcal{H}$ and $\mathcal{S}$ are as defined in §7.3 and the function d_c is an external disturbance.

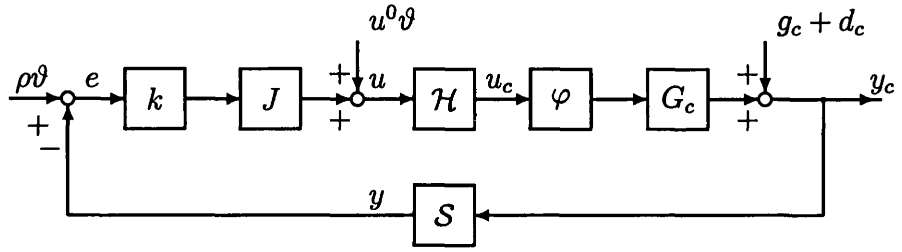

Figure 8.1: Sampled-data low-gain integral control with J integrator

From Figure 8.1 we can derive the following governing equations,

$$y_c = d_c + g_c + G_c(\varphi \circ u_c), \qquad y = \mathcal{S}y_c, \qquad u_c = u^0\vartheta_c + k\mathcal{H}J(\rho\vartheta - y),$$

or, equivalently

$$u_c = u^0\vartheta_c + k\mathcal{H}J(\rho\vartheta - \mathcal{S}(d_c + g_c + G_c(\varphi \circ u_c))). \tag{8.1}$$

Using the linearity of $\mathcal{S}$ and forming the sample-hold discretisation of G_c with generalised sampling, $G := \mathcal{S}G_c\mathcal{H}$, we obtain from (8.1) the corresponding discrete-time equation,

$$u = u^0\vartheta + kJ(\rho\vartheta - (d + g + G(\varphi \circ u))), \tag{8.2}$$

where $g := \mathcal{S}g_c$ and $d := \mathcal{S}d_c$. Equation (8.1) has the unique solution $u_c = \mathcal{H}u$, where $u \in F(\mathbb{Z}_+, \mathbb{R})$ is the unique solution of the discrete-time equation (8.2).

The objective in this subsection is to determine gain parameters k such that the tracking error

$$e_c(t) := \rho - y_c(t) = \rho - (d_c(t) + g_c(t) + (G_c(\varphi \circ u_c))(t)) \tag{8.3}$$

becomes small in a certain sense as $t \to \infty$. For example, we might want to achieve 'tracking in measure', that is, for all $\varepsilon > 0$, the Lebesgue measure of the set $\{t \geq T \mid |e_c(t)| \geq \varepsilon\}$ tends to 0 as $T \to \infty$, or the aim might be 'asymptotic tracking', that is, $\lim_{t\to\infty} e_c(t) = 0$. Tracking in measure is guaranteed if $e_c \in L^p(\mathbb{R}_+, \mathbb{R})$ for some $p \in [1, \infty)$ as seen from Proposition 2.1.13.

Recall that associated with generalised sampling operator $\mathcal{S}$ we have a weighting function $w \in L^2(0,\tau)$ with the property

$$w(s) \geq 0, \qquad \text{a.e. } s \in [0,\tau],$$

where $\tau > 0$ denotes the sampling period. Throughout the remainder of this section we additionally assume that,

$$\int_0^\tau w(s)\,ds = 1.$$

From Theorem 7.4.4, imposing this additional assumption, we obtain

$$\mathbf{G}_c(0) = \mathbf{G}(1), \tag{8.4}$$

where $\mathbf{G}$ denotes the transfer function of $G := \mathcal{S}G_c\mathcal{H}$.

Recall from (5.2) that,

$$f_J(G) := \sup_{q\geq 0}\left\{\operatorname{ess\,inf}_{\theta\in(0,2\pi)}\mathrm{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right]\right\}$$

and, assuming that assumption (A) holds for the transfer function $\mathbf{G}$ of G, we have $-\infty < f_J(G) < -\mathbf{G}(1)/2$ (see Appendix 2, Proposition 12.1.3 (i), for more details).

Theorem 8.1.1. *Let $G_c \in \mathscr{B}(L^2_\alpha(\mathbb{R}_+,\mathbb{R}))$, for some $\alpha < 0$, be a shift-invariant operator with transfer function $\mathbf{G}_c$. Assume that $\mathbf{G}_c(0) > 0$, that $g_c \in L^2_\beta(\mathbb{R}_+,\mathbb{R})$ for some $\beta < 0$, $d_c = d_{c1} + d_{c2}\vartheta_c$ with $d_{c1} \in L^2_\gamma(\mathbb{R}_+,\mathbb{R})$ for some $\gamma < 0$ and $d_{c2} \in \mathbb{R}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous and non-decreasing. Let $\rho \in \mathbb{R}$, assume that $(\rho - d_{c2})/\mathbf{G}_c(0) \in \mathrm{im}\varphi$ and let $u_c : \mathbb{R}_+ \to \mathbb{R}$ be the solution of* (8.1). *Under these conditions the following statements hold.*

1. *Assume that $\varphi - \varphi(0) \in \mathscr{S}(a)$ for some $a \in (0,\infty)$. Then there exists a constant $k^* \in (0,\infty)$ (depending on G_c, φ and ρ) such that for all $k \in (0,k^*)$, the limit $\lim_{t\to\infty} u_c(t) =: u^\infty$ exists and satisfies $\varphi(u^\infty) = (\rho - d_{c2})/\mathbf{G}_c(0)$,*

   $$e_c \in L^2(\mathbb{R}_+,\mathbb{R}) \quad \textit{and} \quad \varphi\circ u_c - \varphi(u^\infty)\vartheta_c \in L^2(\mathbb{R}_+,\mathbb{R}).$$

   Moreover,

   $$\lim_{t\to\infty} e_c(t) = 0,$$

   provided that $\lim_{t\to\infty} g_c(t) = 0$, $\lim_{t\to\infty} d_{c1}(t) = 0$ and G_c satisfies assumption ($\mathrm{B_c}$).

2. *Assume that φ is globally Lipschitz continuous with Lipschitz constant $\lambda > 0$. Then the conclusions of statement 1 are valid with $k^* = 1/|\lambda f_J(G)|$.*

Remarks 8.1.2. (a) Theorem 8.1.1 ensures that the tracking error is square-integrable and hence we have tracking in measure (see Proposition 2.1.13). If $\lim_{t\to\infty} g_c(t) = 0$, $\lim_{t\to\infty} d_{c1}(t) = 0$ and G_c satisfies assumption (B_c), then Theorem 8.1.1 guarantees asymptotic tracking.

(b) Note that in statement 2 of Theorem 8.1.1 the constant k^* depends only on G and the Lipschitz constant of φ, but not on ρ.

(c) A similar comment to Remark 5.1.3 (iv) holds with regard to the assumption $(\rho - d_{c2})/\mathbf{G}_c(0) \in \mathrm{im}\varphi$. $\diamond$

Proof of Theorem 8.1.1. We shall prove Theorem 8.1.1 by applying Theorem 5.1.2 to (8.2). In order to apply Theorem 5.1.2 we need to check that the relevant assumptions are satisfied.

We first note that by Proposition 7.3.2, $G = \mathcal{S}G_c\mathcal{H} \in \mathscr{B}(l^2_\eta(\mathbb{Z}_+, \mathbb{R}))$, for some $\eta \in (0,1)$. Consequently, the transfer function $\mathbf{G}$ of $G = \mathcal{S}G_c\mathcal{H}$ satisfies assumption (A$'$), see Remark 3.4.6 (ii). To see that $g := \mathcal{S}g_c$ satisfies the relevant assumptions we note the following.

Since $g_c \in L^2_\beta(\mathbb{R}_+, \mathbb{R})$ for some $\beta < 0$, Proposition 7.3.2 (ii) yields $\mathcal{S}g_c \in l^2_{\widetilde{\beta}}(\mathbb{Z}_+, \mathbb{R})$ for $\widetilde{\beta} = e^{\beta\tau} \in (0,1)$. So by Remark 5.1.3 (iv) we see that the function $n \mapsto \sum_{j=n}^{\infty}(\mathcal{S}g_c)(j)$, is in $l^2(\mathbb{Z}_+, \mathbb{R})$. Furthermore, we have $\mathcal{S}g_c \in l^1(\mathbb{Z}_+, \mathbb{R})$ implying that $(J\mathcal{S}g_c)(n)$ converges to a finite limit as $n \to \infty$. Hence it follows from Proposition 12.1.5 that $J\mathcal{S}g_c \in m^2(\mathbb{Z}_+, \mathbb{R})$.

To see that $d := \mathcal{S}d_c$ satisfies the relevant assumptions we note the following. A routine calculation shows that $\mathcal{S}\vartheta_c = \vartheta - \delta$. Define $d_1 := \mathcal{S}d_{c1} - d_{c2}\delta$ and $d_2 := d_{c2}$, then clearly $d = d_1 + d_2\vartheta = \mathcal{S}d_c$. Using an argument similar to that which showed $g = \mathcal{S}g_c$ satisfies the relevant assumptions in Theorem 5.1.2, we can also show that d_1 satisfies the required assumptions in Theorem 5.1.2. Finally, note that by (8.4), $(\rho - d_2)/\mathbf{G}(1) = (\rho - d_{c2})/\mathbf{G}_c(0) \in \mathrm{im}\varphi$. Hence the relevant assumptions are satisfied and we may apply Theorem 5.1.2 to (8.2).

Proof of statement 1: With $u_c = \mathcal{H}u$, where $u \in F(\mathbb{Z}_+, \mathbb{R})$ is the unique solution of the discrete-time equation (8.2), an application of Theorem 5.1.2 to (8.2) immediately yields $\lim_{t\to\infty} u_c(t) =: u^\infty$ exists and satisfies $\varphi(u^\infty) = (\rho - d_{c2})/\mathbf{G}_c(0)$. Furthermore, we have that

$$\varphi \circ u - \varphi(u^\infty)\vartheta \in l^2(\mathbb{Z}_+, \mathbb{R}).$$

Consequently, by Proposition 7.3.2 (i),

$$\mathcal{H}((\varphi \circ u) - \varphi(u^\infty)\vartheta) = \mathcal{H}(\varphi \circ u) - \varphi(u^\infty)\mathcal{H}\vartheta = \varphi \circ u_c - \varphi(u^\infty)\vartheta_c \in L^2(\mathbb{R}_+, \mathbb{R}).$$

Since $G_c \in \mathscr{B}(L^2_\alpha(\mathbb{R}_+,\mathbb{R}))$, for some $\alpha < 0$, we have

$$G_c((\varphi \circ u_c) - \varphi(u^\infty)\vartheta_c) \in L^2(\mathbb{R}_+,\mathbb{R}).$$

Note that $g_c \in L^2_\beta(\mathbb{R}_+,\mathbb{R})$ for some $\beta < 0$ and $d_{c1} \in L^2_\gamma(\mathbb{R}_+,\mathbb{R})$ for some $\gamma < 0$, hence $g_c, d_{c1} \in L^2(\mathbb{R}_+,\mathbb{R})$ and

$$g_c + d_{c1} + G_c((\varphi \circ u_c) - \varphi(u^\infty)\vartheta_c) \in L^2(\mathbb{R}_+,\mathbb{R}).$$

Using the linearity of G_c,

$$g_c + d_{c1} + G_c(\varphi \circ u_c) - \varphi(u^\infty)G_c\vartheta_c \in L^2(\mathbb{R}_+,\mathbb{R}),$$

or, equivalently,

$$y_c - d_{c2}\vartheta_c - \varphi(u^\infty)G_c\vartheta_c \in L^2(\mathbb{R}_+,\mathbb{R}). \tag{8.5}$$

Using the fact that $\varphi(u^\infty) = (\rho - d_{c2})/\mathbf{G}_c(0)$ it follows from (8.5) that

$$\mathbf{G}_c(0)y_c - d_{c2}\mathbf{G}_c(0)\vartheta_c - (\rho - d_{c2})G_c\vartheta_c \in L^2(\mathbb{R}_+,\mathbb{R}). \tag{8.6}$$

Noting that,

$$\mathbf{G}_c(0)y_c - \rho G_c\vartheta_c = \mathbf{G}_c(0)(y_c - \rho\vartheta_c) - \rho(G_c\vartheta_c - \mathbf{G}_c(0)\vartheta_c),$$

it follows from (8.6) that

$$\begin{aligned}&\mathbf{G}_c(0)y_c - d_{c2}\mathbf{G}_c(0)\vartheta_c - (\rho - d_{c2})G_c\vartheta_c \\ &\quad = \mathbf{G}_c(0)(y_c - \rho\vartheta_c) + (d_{c2} - \rho)(G_c\vartheta_c - \mathbf{G}_c(0)\vartheta_c) \in L^2(\mathbb{R}_+,\mathbb{R}).\end{aligned} \tag{8.7}$$

Since $G_c \in \mathscr{B}(L^2_\alpha(\mathbb{R}_+,\mathbb{R}))$, assumption ($\mathrm{A}_c$) holds with $\Gamma_c = \mathbf{G}_c(0)$ and $G_c\vartheta_c - \mathbf{G}_c(0)\vartheta_c$, the convolution kernel of the operator $G_c\mathscr{I} - \mathbf{G}_c(0)\mathscr{I} \in \mathscr{B}(L^2(\mathbb{R}_+,\mathbb{R}))$, is in $L^2(\mathbb{R}_+,\mathbb{R})$. Hence from (8.7) we deduce that

$$\mathbf{G}_c(0)(y_c - \rho\vartheta_c) \in L^2(\mathbb{R}_+,\mathbb{R})$$

or, equivalently,

$$e_c = y_c - \rho\vartheta_c \in L^2(\mathbb{R}_+,\mathbb{R}).$$

Assume now that $\lim_{t\to\infty} g_c(t) = 0$, $\lim_{t\to\infty} d_{c1}(t) = 0$ and that (B_c) holds. To see that $\lim_{t\to\infty} e_c(t) = 0$, we first observe that by Lemma 7.2.3,

$$\lim_{t\to\infty}(G_c\vartheta_c)(t) = \mathbf{G}_c(0). \tag{8.8}$$

Noting that $\varphi \circ u - \varphi(u^\infty)\vartheta \in l^2(\mathbb{Z}_+, \mathbb{R})$, we have

$$\lim_{t\to\infty}((\varphi \circ u_c)(t) - \varphi(u^\infty)\vartheta_c(t)) = 0.$$

Since $\varphi \circ u_c - \varphi(u^\infty)\vartheta_c \in PC(\mathbb{R}_+, \mathbb{R}) \cap L^2(\mathbb{R}_+, \mathbb{R})$, invoking assumption ($\mathrm{B}_c$), we obtain

$$\lim_{t\to\infty}((G_c(\varphi \circ u_c))(t) - \varphi(u^\infty)(G_c\vartheta_c)(t)) = 0. \tag{8.9}$$

Combining (8.9), (8.8) and noting that $\varphi(u^\infty) = (\rho - d_{c2})/\mathbf{G}_c(0)$ we have,

$$\lim_{t\to\infty}(G_c(\varphi \circ u_c))(t) = \frac{(\rho - d_{c2})}{\mathbf{G}_c(0)}\mathbf{G}_c(0) = \rho - d_{c2}. \tag{8.10}$$

Combining (8.10) with the additional assumptions

$$\lim_{t\to\infty} g_c(t) = 0, \qquad \lim_{t\to\infty} d_{c1}(t) = 0,$$

it is clear that,

$$\begin{aligned}\lim_{t\to\infty} e_c(t) = \lim_{t\to\infty} y_c(t) - \rho &= \lim_{t\to\infty}(g_c(t) + d_c(t) + (G_c(\varphi \circ u_c))(t)) - \rho \\ &= \rho - \rho \\ &= 0.\end{aligned}$$

Proof of statement 2: This follows as in the proof of statement 2 of Theorem 5.1.2. □

We now change the integrator J to J_0 and consider the feedback system shown in Figure 8.2, where $\xi \in \mathbb{R}$ is the initial state of the integrator.

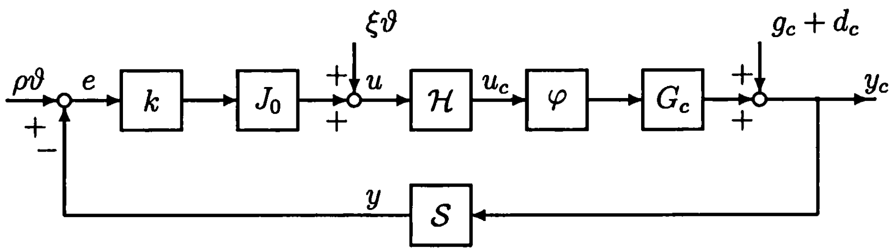

Figure 8.2: Sampled-data low-gain integral control with J_0 integrator

From Figure 8.2 we can derive the following governing equations,

$$y_c = d_c + g_c + G_c(\varphi \circ u_c), \qquad y = \mathcal{S}y_c, \qquad u_c = \xi\vartheta_c + k\mathcal{H}J_0(\rho\vartheta - y),$$

or, equivalently

$$u_c = \xi\vartheta_c + k\mathcal{H}J_0(\rho\vartheta - \mathcal{S}(d_c + g_c + G_c(\varphi \circ u_c))). \tag{8.11}$$

Using the linearity of $\mathcal{S}$ and forming the sample-hold discretisation of G_c with generalised sampling, $G := \mathcal{S}G_c\mathcal{H}$, we obtain from (8.11), the corresponding discrete-time equation,

$$u = \xi\vartheta + kJ_0(\rho\vartheta - (d + g + G(\varphi \circ u))) \tag{8.12}$$

where $g := \mathcal{S}g_c$ and $d := \mathcal{S}d_c$. If $u_c \in F(\mathbb{R}_+, \mathbb{R})$ is a solution of (8.11), then $u(n) := u_c(n\tau)$ for all $n \in \mathbb{Z}_+$, is a solution of (8.12). Conversely, if $u \in F(\mathbb{Z}_+, \mathbb{R})$ is a solution of (8.12), then $u_c = \mathcal{H}u$ is a solution of (8.11).

Recall from (5.12) that,

$$f_{J_0}(G) := \sup_{q \geq 0} \left\{ \operatorname{ess\,inf}_{\theta \in (0,2\pi)} \mathrm{Re} \left[\left(q + \frac{e^{i\theta}}{e^{i\theta - 1}} \right) \mathbf{G}(e^{i\theta}) \right] \right\}$$

and, assuming that assumption (A) holds for the transfer function $\mathbf{G}$ of G, we have $-\infty < f_{J_0}(G) \leq \infty$ (see Appendix 2, Proposition 12.1.4 (ii), for more details).

Theorem 8.1.3. *Let $G_c \in \mathscr{B}(L^2_\alpha(\mathbb{R}_+, \mathbb{R}))$, for some $\alpha < 0$, be a shift-invariant operator with transfer function $\mathbf{G}_c$. Assume that $\mathbf{G}_c(0) > 0$, that $g_c \in L^2_\beta(\mathbb{R}_+, \mathbb{R})$ for some $\beta < 0$, $d_c = d_{c1} + d_{c2}\vartheta_c$ with $d_{c1} \in L^2_\gamma(\mathbb{R}_+, \mathbb{R})$ for some $\gamma < 0$ and $d_{c2} \in \mathbb{R}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous and non-decreasing. Let $\rho \in \mathbb{R}$, assume that $(\rho - d_{c2})/\mathbf{G}_c(0) \in \mathrm{im}\varphi$ and let $u_c : \mathbb{R}_+ \to \mathbb{R}$ be a solution of (8.11). Under these conditions the following statements hold.*

1. *Assume that $\varphi - \varphi(0) \in \mathscr{S}(a)$ for some $a \in (0, \infty)$. Then there exists a constant $k^* \in (0, \infty]$ (depending on G_c, φ and ρ) such that for all $k \in (0, k^*)$, the limit $\lim_{t\to\infty} u_c(t) =: u^\infty$ exists and satisfies $\varphi(u^\infty) = (\rho - d_{c2})/\mathbf{G}_c(0)$,*

   $$e_c \in L^2(\mathbb{R}_+, \mathbb{R}) \quad \textit{and} \quad \varphi \circ u_c - \varphi(u^\infty)\vartheta_c \in L^2(\mathbb{R}_+, \mathbb{R}).$$

   Moreover,

   $$\lim_{t\to\infty} e_c(t) = 0,$$

   provided that $\lim_{t\to\infty} g_c(t) = 0$, $\lim_{t\to\infty} d_{c1}(t) = 0$ and G_c satisfies assumption ($\mathrm{B_c}$). If $f_{J_0}(G) = 0$, then the above conclusions are valid with $k^* = \infty$.

2. *Assume that φ is globally Lipschitz continuous with Lipschitz constant $\lambda > 0$. Then the conclusions of statement 1 are valid with $k^* = 1/|\lambda f_{J_0}(G)|$, where $1/0 := \infty$.*

3. *Under the assumption $f_{J_0}(G) > 0$, the conclusions of statement 1 are valid with $k^* = \infty$.*

Proof. We prove Theorem 8.1.3 by applying Theorem 5.1.4 to (8.12). In order to apply Theorem 5.1.4 we need to check that the relevant assumptions are satisfied. Arguments identical to those in the proof of Theorem 8.1.1 show that the relevant assumptions are satisfied. Statements 1 and 2 follow from the application of Theorem 5.1.4 and the arguments used to prove statements 1 and 2 of Theorem 8.1.1. Statement 3 follows immediately from the application of Theorem 5.1.4. □

Conditions on the weighting function of the generalised sampling operator

We recall that the assumptions made on the weighting function w are

$$w \in L^2(0,\tau) \quad \text{and} \quad \int_0^\tau w(s)\,ds = 1.$$

In this subsection we show that if we impose various extra assumptions on w then, the conditions on g_c and d_c in Theorems 8.1.1 and 8.1.3 can be weakened.

Define $\widetilde{w}$, a periodic extension of w to $\mathbb{R}_+$, by

$$\widetilde{w}(s) := w(s - n\tau), \quad s \in [n\tau, (n+1)\tau), \quad \forall\, n \in \mathbb{Z}_+.$$

Proposition 8.1.4. *If $f \in L^2(\mathbb{R}_+, \mathbb{R})$ is such that $\mathscr{I}(\widetilde{w}f) \in L^2(\mathbb{R}_+, \mathbb{R}) + \mathbb{R}\vartheta_c$, then $J_0 S f \in m^2(\mathbb{Z}_+, \mathbb{R})$.*

Proof. Since $\mathscr{I}(\widetilde{w}f) \in L^2(\mathbb{R}_+, \mathbb{R}) + \mathbb{R}\vartheta_c$, we have $\mathscr{I}(\widetilde{w}f) - \gamma\vartheta_c \in L^2(\mathbb{R}_+, \mathbb{R})$ for some $\gamma \in \mathbb{R}$. Then,

$$\begin{aligned}
(J_0 S f)(n) - \gamma &= \sum_{j=1}^{n} \int_0^\tau \widetilde{w}(s) f((j-1)\tau + s)\,ds - \gamma \\
&= \sum_{j=1}^{n} \int_{(j-1)\tau}^{j\tau} \widetilde{w}(s) f(s)\,ds - \gamma \\
&= \int_0^{n\tau} \widetilde{w}(s) f(s)\,ds - \gamma \\
&= h(n\tau) \\
&= (S_I h)(n), \qquad \forall\, n \geq 1,
\end{aligned}$$

where $h := \mathscr{I}(\widetilde{w}f) - \gamma$. Furthermore, noting that $(\mathcal{S}f)(0) = 0$, it follows that $J_0\mathcal{S}f - \gamma\vartheta = \mathcal{S}_I h$. By assumption $h \in L^2(\mathbb{R}_+, \mathbb{R})$, clearly $\dot{h} = \widetilde{w}f \in L^2(\mathbb{R}_+, \mathbb{R})$ and so $h \in W^{1,2}(\mathbb{R}_+, \mathbb{R})$. An application of Proposition 7.3.2 (iii) yields $\mathcal{S}_I h \in l^2(\mathbb{Z}_+, \mathbb{R})$. Consequently, $J_0\mathcal{S}f - \gamma\vartheta \in l^2(\mathbb{Z}_+, \mathbb{R})$ and we see that $J_0\mathcal{S}f \in m^2(\mathbb{Z}_+, \mathbb{R})$. □

Remark 8.1.5. By Proposition 8.1.4 it follows that the conclusions of Theorems 8.1.1 and 8.1.3 hold under the following assumption on g_c, namely, $g_c \in L^2(\mathbb{R}_+, \mathbb{R})$ and $t \mapsto \int_0^t \widetilde{w}(s)g_c(s)\,ds \in L^2(\mathbb{R}_+, \mathbb{R}) + \mathbb{R}\vartheta_c$. Furthermore, again by Proposition 8.1.4, the conclusions of Theorems 8.1.1 and 8.1.3 hold under the following assumption on d_c, namely, there exists a splitting $d_c = d_{c1} + d_{c2}\vartheta_c$ with $d_{c1} \in L^2(\mathbb{R}_+, \mathbb{R})$ and $d_{c2} \in \mathbb{R}$ such that $t \mapsto \int_0^t \widetilde{w}(s)d_{c1}(s)\,ds \in L^2(\mathbb{R}_+, \mathbb{R}) + \mathbb{R}\vartheta_c$. ◇

For the special case $w \equiv 1/\tau$ we have the following corollary, a trivial consequence of Proposition 8.1.4.

Corollary 8.1.6. *Let $w \equiv 1/\tau$. If $f \in L^2(\mathbb{R}_+, \mathbb{R})$ is such that $\mathscr{I}f \in L^2(\mathbb{R}_+, \mathbb{R}) + \mathbb{R}\vartheta_c$, then $J_0\mathcal{S}f \in m^2(\mathbb{Z}_+, \mathbb{R})$.*

Remark 8.1.7. (a) Let $w \equiv 1/\tau$. Then, by Corollary 8.1.6 Theorems 8.1.1 and 8.1.3 hold under the following assumption on g_c, namely, $g_c \in L^2(\mathbb{R}_+, \mathbb{R})$ and $t \mapsto \int_0^t g_c(s)\,ds \in L^2(\mathbb{R}_+, \mathbb{R}) + \mathbb{R}\vartheta_c$. Furthermore, again by Corollary 8.1.6 Theorems 8.1.1 and 8.1.3 hold under the following assumption on d_c, namely, there exists a splitting $d_c = d_{c1} + d_{c2}\vartheta_c$ with $d_{c1} \in L^2(\mathbb{R}_+, \mathbb{R})$ and $d_{c2} \in \mathbb{R}$ such that $t \mapsto \int_0^t d_{c1}(s)\,ds \in L^2(\mathbb{R}_+, \mathbb{R}) + \mathbb{R}\vartheta_c$.

(b) We remark that if $w \equiv 1/\tau$ the conditions on g_c and d_c in (a) are less restrictive than the corresponding conditions imposed in Theorems 8.1.1 and 8.1.3. To see this, we note that, if $f \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$, then $f \in L^2(\mathbb{R}_+, \mathbb{R})$ and $\int_0^t f(s)\,ds$ converges exponentially fast to $\int_0^\infty f(s)\,ds$ as $t \to \infty$. Consequently,

$$t \mapsto \int_0^t f(s)\,ds \in L^2(\mathbb{R}_+, \mathbb{R}) + \mathbb{R}\vartheta_c.$$

◇

Time-varying gain

Consider the feedback system shown in Figure 8.3, where $\kappa : \mathbb{Z}_+ \to \mathbb{R}$ is a time-varying gain, $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, \mathbb{R}))$ is shift-invariant and φ, ρ, u^0, g_c, d_c are as before.

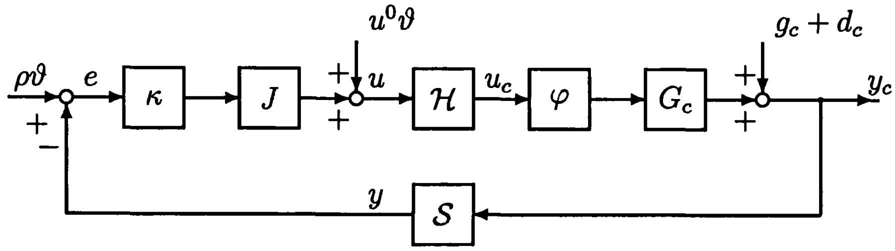

Figure 8.3: Sampled-data time-varying low-gain integral control with J integrator

From Figure 8.3 we can derive the following governing equations,

$$y_c = d_c + g_c + G_c(\varphi \circ u_c), \qquad y = \mathcal{S}y_c, \qquad u_c = u^0\vartheta_c + \mathcal{H}J(\kappa(\rho\vartheta - y)),$$

or, equivalently

$$u_c = u^0\vartheta_c + \mathcal{H}J(\kappa(\rho\vartheta - \mathcal{S}(d_c + g_c + G_c(\varphi \circ u_c)))). \tag{8.13}$$

Using the linearity of $\mathcal{S}$ and forming the sample-hold discretisation of G_c with generalised sampling, $G := \mathcal{S}G_c\mathcal{H}$, we obtain from (8.13), the corresponding discrete-time equation,

$$u = u^0\vartheta + J(\kappa(\rho\vartheta - (d + g + G(\varphi \circ u)))) \tag{8.14}$$

where $g := \mathcal{S}g_c$ and $d := \mathcal{S}d_c$. Equation (8.13) has the unique solution $u_c = \mathcal{H}u$, where $u \in F(\mathbb{Z}_+, \mathbb{R})$ is the unique solution of the discrete-time equation (8.14).

The objective in this subsection is to determine gain functions κ such that the tracking error $e(t)$, defined by (8.3), becomes small in a certain sense as $t \to \infty$.

We impose assumption (A_c) on $\mathbf{G}_c$, the transfer function of G_c, with $\Gamma_c = \mathbf{G}_c(0) := \lim_{s\to 0,\, s\in\mathbb{C}_0} \mathbf{G}_c(s)$. Note that the existence of $\lim_{s\to 0,\, s\in\mathbb{C}_0} \mathbf{G}_c(s)$ is implied by imposing assumption (A_c).

We introduce the set of feasible reference values

$$\mathscr{R}(G_c, \varphi) := \{\mathbf{G}_c(0)v \mid v \in \overline{\mathrm{im}\varphi}\}.$$

It is clear that $\mathscr{R}(G_c, \varphi)$ is an interval provided that φ is continuous. The motivation for the introduction of $\mathscr{R}(G_c, \varphi)$ is as follows. If asymptotic tracking occurs, we would expect that $(\varphi \circ u_c)^\infty := \lim_{t\to\infty}(\varphi \circ u_c)(t)$ exists. Assuming that $(\varphi \circ u_c)^\infty$ is finite and that the final-value theorem holds for the linear system with input-output operator G_c, we may conclude that $\lim_{t\to\infty}(G_c(\varphi \circ u_c))(t) = \mathbf{G}_c(0)(\varphi \circ u_c)^\infty$. If additionally, $\lim_{t\to\infty} g_c(t) = 0$, $\lim_{t\to\infty} d_c(t) = 0$, it follows from (8.3) that $\rho = \mathbf{G}_c(0)(\varphi \circ u_c)^\infty \in \mathscr{R}(G_c, \varphi)$.

Recall from (5.19) that,

$$f_{0,J}(G) := \operatorname{ess\,inf}_{\theta \in (0,2\pi)} \mathrm{Re}\left[\frac{\mathbf{G}(e^{i\theta})}{e^{i\theta}-1}\right]$$

and, assuming that assumption (A) holds for the transfer function $\mathbf{G}$ of G, we have $-\infty < f_{0,J}(G) \leq -\mathbf{G}(1)/2$ (see Appendix 2, Proposition 12.1.3 (ii), for more details).

Theorem 8.1.8. *Let $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, \mathbb{R}))$ be a shift-invariant operator with transfer function $\mathbf{G}_c$. Assume that assumption ($\mathrm{A_c}$) holds with $\mathbf{G}_c(0) > 0$, $g_c \in L^2(\mathbb{R}_+, \mathbb{R})$, $d_c = d_{c1} + d_{c2}\vartheta_c$ with $d_{c1} \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$ and $d_{c2} \in \mathbb{R}$, $\varphi : \mathbb{R} \to \mathbb{R}$ is non-decreasing and globally Lipschitz continuous with Lipschitz constant $\lambda > 0$, $\rho - d_{c2} \in \mathscr{R}(G_c, \varphi)$ and $\kappa : \mathbb{Z}_+ \to \mathbb{R}$ is bounded and non-negative with*

$$\limsup_{n\to\infty} \kappa(n) < 1/|\lambda f_{0,J}(G)|.$$

Let $u_c : \mathbb{R}_+ \to \mathbb{R}$ be the unique solution of (8.13). Then the following statements hold.

1. *The limit $(\varphi \circ u_c)^\infty := \lim_{t\to\infty} \varphi(u_c(t))$ exists and is finite and*

$$\Delta(\varphi \circ u) \in l^2(\mathbb{Z}_+, \mathbb{R}).$$

2. *The signal $y_c = d_c + g_c + G_c(\varphi \circ u_c)$ can be split into $y_c = y_{c1} + y_{c2}$, where y_{c1} has a finite limit satisfying*

$$\lim_{t\to\infty} y_{c1}(t) = \mathbf{G}_c(0)(\varphi \circ u_c)^\infty + d_{c2},$$

and $y_{c2} \in L^2(\mathbb{R}_+, \mathbb{R})$. Under the additional assumptions that

$$\lim_{t\to\infty} g_c(t) = 0, \quad \lim_{t\to\infty} d_{c1}(t) = 0$$

and G_c satisfies assumption ($\mathrm{B_c}$), we have

$$\lim_{t\to\infty} y_{c2}(t) = 0.$$

3. *If $\kappa \notin l^1(\mathbb{Z}_+, \mathbb{R})$, then $\lim_{t\to\infty} y_{c1}(t) = \rho$ and the error signal e_c can be split into $e_c = e_{c1} + e_{c2}$, where $\lim_{t\to\infty} e_{c1}(t) = 0$ and $e_{c2} \in L^2(\mathbb{R}_+, \mathbb{R})$.*

4. *If $\kappa \notin l^1(\mathbb{Z}_+, \mathbb{R})$,*

$$\lim_{t\to\infty} g_c(t) = 0, \quad \lim_{t\to\infty} d_{c1}(t) = 0$$

and G_c satisfies assumption ($\mathrm{B_c}$), *then*

$$\lim_{t\to\infty} e_c(t) = 0.$$

5. *If $\rho - d_{c2}$ is an interior point of $\mathscr{R}(G_c,\varphi)$, then u_c is bounded.*

Remarks 8.1.9. (a) It follows from Proposition 2.1.13 that statement 3 of Theorem 8.1.8 implies tracking in measure. Under the additional assumptions in statement 4 of Theorem 8.1.8 we have asymptotic tracking.

(b) Note that it is not necessary to know $f_{0,J}(G)$ or the constant λ in order to apply Theorem 8.1.8. If κ is chosen such that $\kappa(n) \to 0$ as $n \to \infty$ and $\kappa \notin l^1(\mathbb{Z}_+,\mathbb{R})$ (e.g., $\kappa(n) = (1+n)^{-p}$ with $p \in (0,1]$), then the conclusions of statements 3 and 4 hold. However, from a practical point of view, gain functions κ with $\lim_{n\to\infty}\kappa(n) = 0$ might not be appropriate, since the system essentially operates in open loop as $n \to \infty$.

(c) If $\rho - d_{c2}$ is not an interior point of $\mathscr{R}(G_c,\varphi)$ then u_c might be unbounded. A trivial example is given by $\varphi = \arctan$, $\rho = (1/2)\mathbf{G}_c(0)\pi$, $d_{c2} = 0$ and $\kappa \notin l^1(\mathbb{Z}_+,\mathbb{R})$, in which case it follows from statement 3 that $(1/2)\pi = (\varphi\circ u_c)^\infty$ and hence $\lim_{t\to\infty} u_c(t) = \infty$.

(d) A similar comment to Remark 5.2.3 (iii) holds with regard to the assumption $\rho - d_{c2} \in \mathscr{R}(G_c,\varphi)$. ◇

Proof of Theorem 8.1.8. We shall prove Theorem 8.1.8 by applying Theorem 5.2.2, with $\psi = \mathrm{id}$, to (8.14). In order to apply Theorem 5.2.2 we need to check that the relevant assumptions are satisfied.

We first note that if $\mathbf{G}_c$ satisfies assumption ($\mathrm{A_c}$) then, by Theorem 7.4.4, $\mathbf{G}$, the transfer function of $G = \mathcal{S}G_c\mathcal{H}$, satisfies assumption (A). Furthermore, by (8.4), $\mathscr{R}(G,\varphi) = \mathscr{R}(G_c,\varphi)$. Since by assumption $g_c \in L^2(\mathbb{R}_+,\mathbb{R})$, it follows from Proposition 7.3.2 (ii) that $g := \mathcal{S}g_c \in l^2(\mathbb{Z}_+,\mathbb{R})$. To see that $d := \mathcal{S}d_c$ satisfies the relevant assumptions we note the following. A routine calculation shows that $\mathcal{S}\vartheta_c = \vartheta - \delta$. Define $d_1 := \mathcal{S}d_{c1} - d_{c2}\delta$ and $d_2 := d_{c2}$, then clearly $d = d_1 + d_2\vartheta = \mathcal{S}d_c$. Since $d_{c1} \in L^2_\alpha(\mathbb{R}_+,\mathbb{R})$ for some $\alpha < 0$, Proposition 7.3.2 (ii) yields $\mathcal{S}d_{c1} \in l^2_\beta(\mathbb{Z}_+,\mathbb{R})$ for $\beta = e^{\alpha\tau} \in (0,1)$ and consequently $d_1 \in l^2_\beta(\mathbb{Z}_+,\mathbb{R})$. So by Remark 5.1.3 (iv) we see that the function $n \mapsto \sum_{j=n}^\infty |d_1(n)|$ is in $l^2(\mathbb{Z}_+,\mathbb{R})$. Furthermore, since $d_{c1} \in L^2_\alpha(\mathbb{R}_+,\mathbb{R})$ for some $\alpha < 0$, $\mathcal{S}d_{c1} \in l^1(\mathbb{Z}_+,\mathbb{R})$ and it follows from the definition of δ that $d_1 \in l^1(\mathbb{Z}_+,\mathbb{R})$ and hence d_1 satisfies the required assumptions. Hence the relevant assumptions are satisfied and we may apply Theorem 5.2.2 to (8.14).

Statements 1 and 5 follow immediately from the application of Theorem 5.2.2 and the fact that $u_c = \mathcal{H}u$.

Proof of Statement 2: Note that

$$y_c = d_c + g_c + G_c(\varphi \circ u_c).$$

By statement 1, it follows from Proposition 7.3.4, with $v = \varphi \circ u$, that

$$G_c(\mathcal{H}(\varphi \circ u)) = G_c(\varphi \circ u_c) = k_{c1} + k_{c2}$$

where $\lim_{t\to\infty} k_{c1}(t) = \mathbf{G}_c(0)(\varphi \circ u_c)^\infty$ and $k_{c2} \in L^2(\mathbb{R}_+, \mathbb{R})$. Consequently, setting $y_{c1} := d_{c2}\vartheta_c + k_{c1}$ and $y_{c2} := d_{c1} + g_c + k_{c2}$, it follows that $y_c = y_{c1} + y_{c2}$ and $\lim_{t\to\infty} y_{c1}(t) = \mathbf{G}_c(0)(\varphi \circ u_c)^\infty + d_{c2}$. Since, by assumption $d_{c1} \in L^2_\alpha(\mathbb{R}_+, \mathbb{R}) \subset L^2(\mathbb{R}_+, \mathbb{R})$ and $g_c \in L^2(\mathbb{R}_+, \mathbb{R})$, we have $y_{c2} \in L^2(\mathbb{R}_+, \mathbb{R})$. Under the additional assumptions, $\lim_{t\to\infty} g_c(t) = 0$, $\lim_{t\to\infty} d_{c1}(t) = 0$ and (B$_c$), it is clear from Proposition 7.3.4 that $\lim_{t\to\infty} y_{c2}(t) = 0$.

Proof of Statement 3: By statements 2 and 3 of Theorem 5.2.2 we have that

$$\rho = \lim_{n\to\infty} y(n) = \mathbf{G}(1)(\varphi \circ u)^\infty + d_2, \tag{8.15}$$

where $y = \mathcal{S}y_c$. Using statement 2, the fact that $u_c = \mathcal{H}u$, (8.4) and (8.15), we obtain

$$\begin{aligned}
\lim_{t\to\infty} y_{c1}(t) &= \mathbf{G}_c(0) \lim_{t\to\infty} (\varphi \circ u_c)(t) + d_{c2} \\
&= \mathbf{G}_c(0)(\varphi \circ u_c)^\infty + d_{c2} \\
&= \mathbf{G}(1)(\varphi \circ u)^\infty + d_2 \\
&= \rho.
\end{aligned} \tag{8.16}$$

Finally, setting $e_{c1} = \rho - y_{c1}$ and $e_{c2} = -y_{c2}$, we obtain the splitting

$$e_c = \rho - y_c = e_{c1} + e_{c2}.$$

It follows immediately from (8.16) that $\lim_{t\to\infty} e_{c1}(t) = 0$ and by statement 2 that $e_{c2} \in L^2(\mathbb{R}_+, \mathbb{R})$.

Proof of Statement 4: If it is additionally assumed that (B$_c$) holds, $\lim_{t\to\infty} g_c(t) = 0$ and $\lim_{t\to\infty} d_{c1}(t) = 0$, it follows from statement 2 that $\lim_{t\to\infty} e_{c2}(t) = 0$, implying that $\lim_{t\to\infty} e_c(t) = 0$. □

We now change the integrator J to J_0 and consider the feedback system shown in Figure 8.4 where $\xi \in \mathbb{R}$ is the initial state of the integrator.

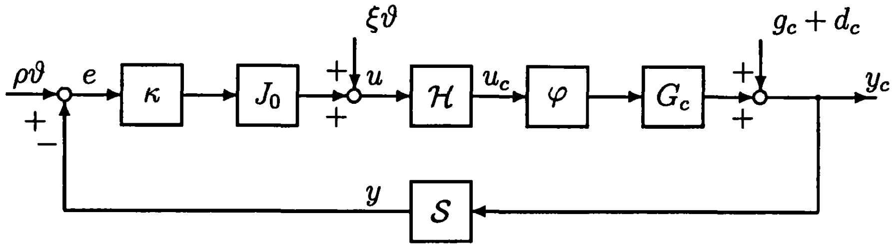

Figure 8.4: Sampled-data time-varying low-gain integral control with J_0 integrator

From Figure 8.4 we can derive the following governing equations,

$$y_c = d_c + g_c + G_c(\varphi \circ u_c), \qquad y = \mathcal{S}y_c, \qquad u_c = \xi\vartheta_c + \mathcal{H}J_0(\kappa(\rho\vartheta - y)),$$

or, equivalently

$$u_c = \xi\vartheta_c + \mathcal{H}J_0(\kappa(\rho\vartheta - \mathcal{S}(d_c + g_c + G_c(\varphi \circ u_c)))). \tag{8.17}$$

Using the linearity of $\mathcal{S}$ and forming the sample-hold discretisation of G_c with generalised sampling, $G := \mathcal{S}G_c\mathcal{H}$, we obtain from (8.17), the corresponding discrete-time equation,

$$u = \xi\vartheta + J_0(\kappa(\rho\vartheta - (d + g + G(\varphi \circ u)))) \tag{8.18}$$

where $g := \mathcal{S}g_c$ and $d := \mathcal{S}d_c$. If $u_c \in F(\mathbb{R}_+, \mathbb{R})$ is a solution of (8.17), then $u(n) := u_c(n\tau)$ for all $n \in \mathbb{Z}_+$, is a solution of (8.18). Conversely, if $u \in F(\mathbb{Z}_+, \mathbb{R})$ is a solution of (8.18), then $u_c = \mathcal{H}u$ is a solution of (8.17).

Recall from (5.33) that,

$$f_{0,J_0}(G) := \operatorname{ess\,inf}_{\theta \in (0,2\pi)} \mathrm{Re}\left[\frac{\mathbf{G}(e^{i\theta})e^{i\theta}}{e^{i\theta} - 1}\right]$$

and, assuming that assumption (A) holds for the transfer function $\mathbf{G}$ of G, we have $-\infty < f_{0,J_0}(G) \leq \mathbf{G}(\infty) - \mathbf{G}(1)/2$, where $\mathbf{G}(\infty) := \lim_{|z| \to \infty, z \in \mathbf{E}_1} \mathbf{G}(z)$ (see Appendix 2, Proposition 12.1.4 (ii), for more details).

Theorem 8.1.10. *Let $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, \mathbb{R}))$ be a shift-invariant operator with transfer function $\mathbf{G}_c$. Assume that assumption* ($\mathrm{A_c}$) *holds with $\mathbf{G}_c(0) > 0$, $g_c \in L^2(\mathbb{R}_+, \mathbb{R})$, $d_c = d_{c1} + d_{c2}\vartheta_c$ with $d_{c1} \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$ and $d_{c2} \in \mathbb{R}$, $\varphi : \mathbb{R} \to \mathbb{R}$ is non-decreasing and globally Lipschitz continuous with Lipschitz* *constant $\lambda > 0$, $\rho - d_{c2} \in \mathscr{R}(G_c, \varphi)$ and $\kappa : \mathbb{Z}_+ \to \mathbb{R}$ is bounded and non-negative with*

$$\limsup_{n \to \infty} \kappa(n) < 1/|\lambda f_{0,J_0}(G)|,$$

if $f_{0,J_0}(G) \leq 0$, where $1/0 := \infty$. Let $u_c : \mathbb{R}_+ \to \mathbb{R}$ be a solution of (8.17). *Then statements 1-5 of Theorem* 8.1.8 *hold.*

Proof. We prove Theorem 8.1.10 by applying Theorem 5.2.4, with ψ = id, to (8.18). In order to apply Theorem 5.2.4 we need to check that the relevant assumptions are satisfied. Arguments identical to those in the proof of Theorem 8.1.8 show that the relevant assumptions are satisfied. Statements 1-5 now follow from the application of Theorem 5.2.4 and arguments identical to those used to prove Theorem 8.1.8. □

8.2 Sampled-data low-gain integral control in the presence of input and output nonlinearities

Consider the feedback system shown in Figure 8.5, where κ, G_c, φ, ρ, u^0, g_c, d_c are as before and additionally $\psi : \mathbb{R} \to \mathbb{R}$ is an output non-linearity.

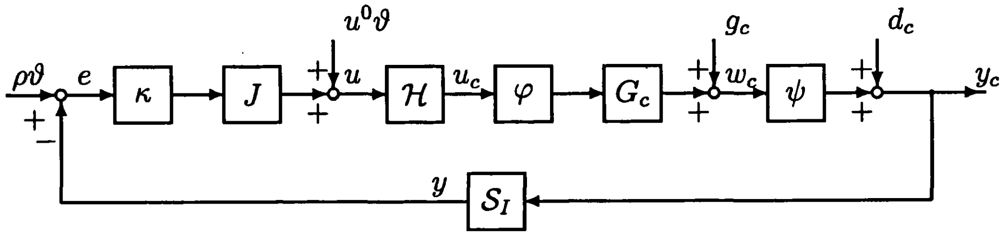

Figure 8.5: Sampled-data time-varying low-gain integral control with input and output non-linearities and J integrator

In §8.1 we only considered feedback systems with input non-linearities. In the general setting, where the shift-invariant input-output operator $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, \mathbb{R}))$, it seems difficult to obtain generalisations of the time-varying gain results in §8.1 which encompass static output non-linearities ψ. The reason for this is that, in general, the operator $\mathcal{S}$ does not commute with non-linearities. If however, we restrict to considering input-output operators G_c whose impulse response is a finite Borel measure and replace $\mathcal{S}$ with $\mathcal{S}_I$, then the inclusion of static output non-linearities is feasible as seen from the following proposition.

Proposition 8.2.1. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous static non-linearity. Then,*

$$\psi \circ (\mathcal{S}_I u) = \mathcal{S}_I(\psi \circ u), \qquad \forall\, u \in F(\mathbb{R}_+, \mathbb{R}).$$

Proof. Let $u \in F(\mathbb{R}_+, \mathbb{R})$. Then,

$$\psi((\mathcal{S}_I u)(n)) = \psi(u(n\tau)) = (\psi \circ u)(n\tau) = (\mathcal{S}_I(\psi \circ u))(n), \qquad \forall\, n \in \mathbb{Z}_+.$$

□

From Figure 8.5 we can derive the following governing equations,

$$w_c = g_c + G_c(\varphi \circ u_c), \quad y_c = \psi \circ w_c + d_c, \quad y = \mathcal{S}_I y_c, \quad u_c = u^0 \vartheta_c + \mathcal{H}J(\kappa(\rho\vartheta - y)),$$

or, equivalently

$$u_c = u^0 \vartheta_c + \mathcal{H}J(\kappa(\rho\vartheta - \mathcal{S}_I(d_c + \psi(g_c + G_c(\varphi \circ u_c))))). \tag{8.19}$$

Using the linearity of $\mathcal{S}_I$, invoking Proposition 8.2.1 and forming the sample-hold discretisation of G_c with idealised sampling, $G := \mathcal{S}_I G_c \mathcal{H}$, we obtain from (8.19), the corresponding discrete-time equation,

$$u = u^0 \vartheta + J(\kappa(\rho\vartheta - d - \psi(g + G(\varphi \circ u)))) \tag{8.20}$$

where $g := \mathcal{S}_I g_c$ and $d := \mathcal{S}_I d_c$. Equation (8.19) has the unique solution $u_c = \mathcal{H}u$, where $u \in F(\mathbb{Z}_+, \mathbb{R})$ is the unique solution of the discrete-time equation (8.20).

The objective in this section is to determine gain functions κ such that the tracking error

$$e_c(t) := \rho - y_c(t) = \rho - d_c(t) - \psi(g_c(t) + (G_c(\varphi \circ u_c))(t)),$$

becomes small in a certain sense as $t \to \infty$. The set of feasible reference values is now defined by

$$\mathscr{R}(G_c, \varphi, \psi) := \{\psi(\mathbf{G}_c(0)v) \mid v \in \overline{\operatorname{im}\varphi}\}.$$

It is clear that $\mathscr{R}(G_c, \varphi, \psi)$ is an interval provided that φ and ψ are continuous.

Theorem 8.2.2. *Let G_c be a convolution operator, the kernel of which is a finite Borel measure on $\mathbb{R}_+$. Denote the transfer function of G_c by $\mathbf{G}_c$. Assume that, assumption* (A$_c$) *holds with $\mathbf{G}_c(0) > 0$, the function $g_c \in L^2(\mathbb{R}_+, \mathbb{R})$ is such that $\mathcal{S}_I g_c \in l^2(\mathbb{Z}_+, \mathbb{R})$, $d_c = d_{c1} + d_{c2}\vartheta_c$ with $d_{c1} \in L^2(\mathbb{R}_+, \mathbb{R})$, $\mathcal{S}_I d_{c1} \in l^1(\mathbb{Z}_+, \mathbb{R})$, $n \mapsto \sum_{j=n}^{\infty} |d_{c1}(j\tau)| \in l^2(\mathbb{Z}_+, \mathbb{R})$ and $d_{c2} \in \mathbb{R}$, $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are non-decreasing and globally Lipschitz continuous with Lipschitz constants $\lambda_1 > 0$ and $\lambda_2 > 0$, $\rho - d_{c2} \in \mathscr{R}(G_c, \varphi, \psi)$ and $\kappa : \mathbb{Z}_+ \to \mathbb{R}$ is bounded and non-negative* *with*

$$\limsup_{n\to\infty} \kappa(n) < 1/|\lambda_1\lambda_2 f_{0,J}(G)|.$$

Let $u_c : \mathbb{R}_+ \to \mathbb{R}$ *be the unique solution of* (8.19). *Then the following statements hold.*

1. *The limit* $(\varphi \circ u_c)^\infty := \lim_{t\to\infty} \varphi(u_c(t))$ *exists and is finite and*

$$\triangle(\varphi \circ u) \in l^2(\mathbb{Z}_+, \mathbb{R}).$$

2. *The signals* $w_c = g_c + G_c(\varphi \circ u_c)$ *and* $y_c = \psi \circ w_c + d_c$ *can be split into* $w_c = w_{c1} + w_{c2}$ *and* $y_c = y_{c1} + y_{c2}$*, where* w_{c1} *and* y_{c1} *have finite limits satisfying*

$$\lim_{t\to\infty} w_{c1}(t) = \mathbf{G}_c(0)(\varphi \circ u_c)^\infty, \quad \lim_{t\to\infty} y_{c1}(t) = \psi(\mathbf{G}_c(0)(\varphi \circ u_c)^\infty) + d_{c2},$$

and $w_{c2}, y_{c2} \in L^2(\mathbb{R}_+, \mathbb{R})$.
Under the additional assumption that $\lim_{t\to\infty} g_c(t) = 0$ *we have*

$$\lim_{t\to\infty} w_{c2}(t) = 0.$$

If further, $\lim_{t\to\infty} d_{c1}(t) = 0$ *then,* $\lim_{t\to\infty} y_{c2}(t) = 0$.

3. *If* $\kappa \notin l^1(\mathbb{Z}_+, \mathbb{R})$*, then* $\lim_{t\to\infty} y_{c1}(t) = \rho$ *and the error signal* e_c *can be split into* $e_c = e_{c1} + e_{c2}$*, where* $\lim_{t\to\infty} e_1(t) = 0$ *and* $e_{c2} \in L^2(\mathbb{R}_+, \mathbb{R})$.

4. *If* $\kappa \notin l^1(\mathbb{Z}_+, \mathbb{R})$ *and*

$$\lim_{t\to\infty} g_c(t) = 0, \quad \lim_{t\to\infty} d_{c1}(t) = 0,$$

we have

$$\lim_{t\to\infty} e_c(t) = 0.$$

5. *If* $\rho - d_{c2}$ *is an interior point of* $\mathscr{R}(G_c, \varphi, \psi)$*, then* u_c *is bounded.*

Remarks 8.2.3. (a) Statements (a)-(c) of Remarks 8.1.9 (or suitable modifications thereof) remain valid in the context of Theorem 8.2.2.

(b) Assume additionally that g_c is bounded. Then the assumption that $S_I g_c \in l^2(\mathbb{Z}_+, \mathbb{R})$ is satisfied if there exists $\varepsilon > 0$ such that $g_c(t) = O(e^{-\varepsilon t})$ as $t \to \infty$, or, if there exists $T \geq 0$ such that $g_c \in W^{1,2}([T, \infty), \mathbb{R})$. ◇

Proof of Theorem 8.2.2. We shall prove Theorem 8.2.2 by applying Theorem 5.2.2 to (8.20). In order to apply Theorem 5.2.2 we need to check that the relevant assumptions are satisfied.

We first note that if $\mathbf{G}_c$ satisfies assumption (A_c), then by Theorem 7.4.9, $\mathbf{G}$ the transfer function of $G = \mathcal{S}_I G_c \mathcal{H}$ satisfies assumption (A). Furthermore, by Theorem 7.4.9, $\mathscr{R}(G, \varphi, \psi) = \mathscr{R}(G_c, \varphi, \psi)$. It is clear that $g := \mathcal{S}_I g_c$ and $d := \mathcal{S}_I d_c$ satisfy the relevant assumptions in Theorem 5.2.2. Hence we may apply Theorem 5.2.2 to (8.20).

Statements 1 and 5 follow immediately from the application of Theorem 5.2.2 and the fact that $u_c = \mathcal{H}u$.

Proof of Statement 2: Note that

$$w_c = g_c + G_c(\varphi \circ u_c).$$

By statement 1, it follows from Proposition 7.3.4, with $v = \varphi \circ u$, that

$$G_c(\mathcal{H}(\varphi \circ u)) = G_c(\varphi \circ u_c) = k_{c1} + k_{c2}$$

where $\lim_{t\to\infty} k_{c1}(t) = \mathbf{G}_c(0)(\varphi \circ u_c)^\infty$ and $k_{c2} \in L^2(\mathbb{R}_+, \mathbb{R})$. Consequently, setting $w_{c1} := k_{c1}$ and $w_{c2} := g_c + k_{c2}$ it follows that $w_c = w_{c1} + w_{c2}$ and $\lim_{t\to\infty} w_{c1}(t) = \mathbf{G}_c(0)(\varphi \circ u_c)^\infty$. Since, by assumption, $g_c \in L^2(\mathbb{R}_+, \mathbb{R})$ we have $w_{c2} \in L^2(\mathbb{R}_+, \mathbb{R})$. Setting $y_{c1} := \psi \circ w_{c1} + d_{c2}$ and $y_{c2} := y_c - y_{c1} = \psi \circ w_c - \psi \circ w_{c1}$, we obtain $y_c = y_{c1} + y_{c2}$ and y_{c1} has limit $\lim_{t\to\infty} y_{c1}(t) = \psi(\mathbf{G}_c(0)(\varphi \circ u)^\infty) + d_{c2}$. Furthermore, $y_{c2} \in L^2(\mathbb{R}_+, \mathbb{R})$ because of the global Lipschitz continuity of ψ and the facts that $w_{c2} \in L^2(\mathbb{R}_+, \mathbb{R})$ and $d_{c1} \in L^2(\mathbb{R}_+, \mathbb{R})$. Under the additional assumption that $\lim_{t\to\infty} g_c(t) = 0$ and noting that assumption (B_c) holds for G_c, (see Remark 7.3.3), it follows from Proposition 7.3.4 that $\lim_{t\to\infty} w_{c2}(t) = 0$. Consequently, if we further assume that $\lim_{t\to\infty} d_{c1}(t) = 0$, we may conclude that $\lim_{t\to\infty} y_{c2}(t) = 0$, completing the proof of statement 2.

Proof of Statement 3: By statements 2 and 3 of Theorem 5.2.2 we have that

$$\rho = \lim_{n\to\infty} y(n) = \psi(\mathbf{G}(1)(\varphi \circ u)^\infty) + d_{c2}. \tag{8.21}$$

Using statement 2, Theorem 7.4.9 and (8.21), we obtain

$$\begin{aligned} \lim_{t\to\infty} y_{c1}(t) &= \psi(\mathbf{G}_c(0)(\varphi \circ u_c)^\infty) + d_{c2} \\ &= \psi(\mathbf{G}(1)(\varphi \circ u)^\infty) + d_{c2} \\ &= \rho. \end{aligned} \tag{8.22}$$

Finally, setting $e_{c1} = \rho - y_{c1}$ and $e_{c2} = -y_{c2}$, we obtain the splitting

$$e_c = \rho - y_c = e_{c1} + e_{c2}.$$

It follows immediately from (8.22) that $\lim_{t\to\infty} e_{c1}(t) = 0$ and by statement 2 that $e_{c2} \in L^2(\mathbb{R}_+, \mathbb{R})$.

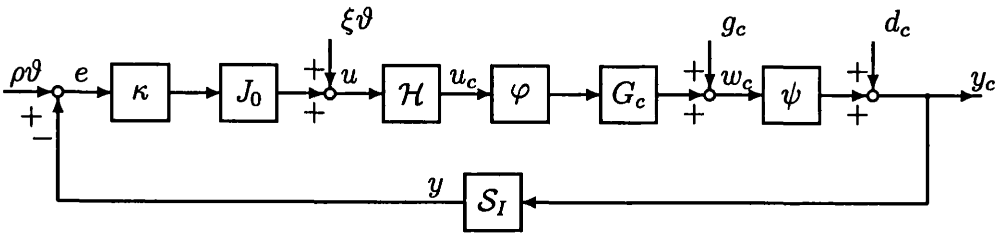

Figure 8.6: Sampled-data time-varying low-gain integral control with input and output non-linearities and J_0 integrator

Proof of Statement 4: Under the additional assumptions $\lim_{t\to\infty} g_c(t) = 0$ and $\lim_{t\to\infty} d_{c1}(t) = 0$, it follows from statement 2 that $\lim_{t\to\infty} e_{c2}(t) = 0$, implying that $\lim_{t\to\infty} e_c(t) = 0$. □

We now change the integrator J to J_0 and consider the feedback system shown in Figure 8.6 where $\xi \in \mathbb{R}$ is the initial state of the integrator. From Figure 8.6 we can derive the following governing equations,

$$w_c = g_c + G_c(\varphi \circ u_c), \quad y_c = \psi \circ w_c + d_c, \quad y = \mathcal{S}_I y_c, \quad u_c = \xi\vartheta_c + \mathcal{H}J_0(\kappa(\rho\vartheta - y)),$$

or, equivalently

$$u_c = \xi\vartheta_c + \mathcal{H}J_0(\kappa(\rho\vartheta - \mathcal{S}_I(d_c + \psi(g_c + G_c(\varphi \circ u_c))))). \tag{8.23}$$

Using the linearity of $\mathcal{S}_I$, Proposition 8.2.1 and forming the sample-hold discretisation of G_c with idealised sampling, $G := \mathcal{S}_I G_c \mathcal{H}$, we obtain from (8.23), the corresponding discrete-time equation,

$$u = \xi\vartheta + J_0(\kappa(\rho\vartheta - d - \psi(g + G(\varphi \circ u)))) \tag{8.24}$$

where $g := \mathcal{S}_I g_c$ and $d := \mathcal{S}_I d_c$. If $u_c \in F(\mathbb{R}_+, \mathbb{R})$ is a solution of (8.23), then $u(n) := u_c(n\tau)$ for all $n \in \mathbb{Z}_+$, is a solution of (8.24). Conversely, if $u \in F(\mathbb{Z}_+, \mathbb{R})$ is a solution of (8.24), then $u_c = \mathcal{H}u$ is a solution of (8.23).

Theorem 8.2.4. *Let G_c be a convolution operator, the kernel of which is a finite Borel measure on $\mathbb{R}_+$. Denote the transfer function of G_c by $\mathbf{G}_c$. Assume that, assumption* (A_c) *holds with $\mathbf{G}_c(0) > 0$, the function $g_c \in L^2(\mathbb{R}_+, \mathbb{R})$ is such that $\mathcal{S}_I g_c \in l^2(\mathbb{Z}_+, \mathbb{R})$, $d_c = d_{c1} + d_{c2}\vartheta_c$ with $d_{c1} \in L^2(\mathbb{R}_+, \mathbb{R})$, $\mathcal{S}_I d_{c1} \in l^1(\mathbb{Z}_+, \mathbb{R})$, $n \mapsto \sum_{j=n}^{\infty} |d_{c1}(j\tau)| \in l^2(\mathbb{Z}_+, \mathbb{R})$ and $d_{c2} \in \mathbb{R}$, $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are non-decreasing and globally Lipschitz continuous with Lipschitz constants $\lambda_1 > 0$ and $\lambda_2 > 0$, $\rho - d_{c2} \in \mathscr{R}(G_c, \varphi, \psi)$ and $\kappa : \mathbb{Z}_+ \to \mathbb{R}$ is bounded and non-negative with*

$$\limsup_{n\to\infty} \kappa(n) < 1/|\lambda_1 \lambda_2 f_{0,J_0}(G)|,$$

if $f_{0,J_0}(G) \leq 0$, where $1/0 := \infty$. Let $u_c : \mathbb{R}_+ \to \mathbb{R}$ be a solution of (8.23). *Then statements 1-5 of Theorem* 8.2.2 *hold.*

Proof. We prove Theorem 8.2.4 by applying Theorem 5.2.4 to (8.24). In order to apply Theorem 5.2.4 we need to check that the relevant assumptions are satisfied. Arguments identical to those in the proof of Theorem 8.2.2 show that the relevant assumptions are satisfied. Statements 1-5 now follow from the application of Theorem 5.2.4 and arguments identical to those used to prove Theorem 8.2.2. □

8.3 Notes and references

In the recent paper [9] an input-output approach to continuous-time low-gain integral control of L^2 stable linear infinite-dimensional systems subject to actuator and sensor non-linearities was developed. In this chapter we have extended the approach in [9] to the sampled-data integral control schemes shown in Figures 8.1-8.6. In particular, the results in this chapter can be considered as substantial and far reaching generalisations of the results in [17] which were obtained in a finite-dimensional state-space setting. The main results on integral control, Theorems 8.1.1, 8.1.3, 8.1.8, 8.1.10, 8.2.2 and 8.2.4, are new and form the basis of [6] and [7]. Note that the results in this chapter on integral control show that tracking of feasible constant reference values is still achievable even in the presence of certain output disturbances. We emphasize that in contrast to the results in [17], which were obtained for finite-dimensional exponentially stable state-space systems, Theorems 8.1.1, 8.1.3, 8.1.8, 8.1.10, 8.2.2 and 8.2.4 apply to infinite-dimensional well-posed state-space systems (see Chapter 9).

Chapter 9

Sampled-data control of infinite-dimensional well-posed state-space systems

9.1 Well-posed state-space systems

In this chapter we apply the results in Chapter 8 to well-posed state-space systems. There are a number of equivalent definitions of well-posed systems, see [12], [52], [53], [54], [55], [56], [57], [60], [63] and [65]. We will be brief in the following and refer the reader to the above references for more details. Throughout this chapter, we shall be considering a well-posed system Σ with state-space X, input space $\mathbb{R}$ and output space $\mathbb{R}$, generating operators (A, B, C), input-output operator G_c and transfer function $\mathbf{G}_c$. Here X is a real Hilbert space with norm denoted by $\|\cdot\|$, A is the generator of a strongly continuous semigroup $\mathbf{T} = (\mathbf{T}_t)_{t\geq 0}$ on X, $B \in \mathscr{B}(\mathbb{R}, X_{-1})$ and $C \in \mathscr{B}(X_1, \mathbb{R})$, where X_1 denotes the space dom(A) endowed with the norm $\|x\|_1 := \|x\| + \|Ax\|$ (the graph norm of A), whilst X_{-1} denotes the completion of X with respect to the norm $\|x\|_{-1} = \|(\alpha I - A)^{-1}x\|$, where $\alpha \in \text{res}(A)$ (different choices of α lead to equivalent norms). Clearly, $X_1 \subset X \subset X_{-1}$ and the canonical injections are bounded and dense. The semigroup $\mathbf{T}$ restricts to a strongly continuous semigroup on X_1 and extends to a strongly continuous semigroup on X_{-1} with the exponential growth constant being the same on all three spaces. Correspondingly, A restricts to a generator on X_1 and extends to a generator on X_{-1}. We shall use the same symbol $\mathbf{T}$ (respectively, A) for the original semigroup (respectively, generator) and the associated restrictions and extensions: with this convention, we may write $A \in \mathscr{B}(X, X_1)$ (considered as a generator on X_{-1}, the domain of A is X). Moreover, B is an *admissible control operator* for $\mathbf{T}$ and C is an *admissible observation operator* for

$\mathbf{T}$, that is, for each $t \in \mathbb{R}_+$ there exist $\alpha_t \geq 0$ and $\beta_t \geq 0$ such that

$$\left\| \int_0^t \mathbf{T}_{t-s} B v(s)\, ds \right\| \leq \alpha_t \|v\|_{L^2([0,t])}, \qquad \forall\, v \in L^2([0,t])$$

and

$$\left(\int_0^t \|C\mathbf{T}_s z\|^2\, ds \right)^{1/2} \leq \beta_t \|z\|, \qquad \forall\, z \in X_1.$$

The control operator B is said to be *bounded* if it is so as a map from the input space $\mathbb{R}$ to the state-space X, otherwise B is said to be *unbounded.* The observation operator C is said to be *bounded* if it can be extended continuously to X, otherwise, C is said to be *unbounded.*

The so-called Λ-*extension* C_Λ of C is defined by

$$C_\Lambda z := \lim_{s \to \infty, s \in \mathbb{R}} Cs(sI - A)^{-1} z,$$

with $\mathrm{dom}(C_\Lambda)$ consisting of all $z \in X$ for which the above limit exists. For every $z \in X$, $\mathbf{T}_t z \in \mathrm{dom}(C_\Lambda)$ for a.a. $t \in \mathbb{R}_+$ and, if $\omega > \omega(\mathbf{T})$, then $C_\Lambda \mathbf{T} z \in L^2_\omega(\mathbb{R}_+, \mathbb{R})$, where

$$\omega(\mathbf{T}) := \lim_{t \to \infty} \frac{1}{t} \ln \|\mathbf{T}_t\|$$

denotes the exponential growth constant of $\mathbf{T}$. The transfer function $\mathbf{G}_c$ satisfies

$$\frac{1}{s - s_0}(\mathbf{G}_c(s) - \mathbf{G}_c(s_0)) = -C(sI - A)^{-1}(s_0 I - A)^{-1} B,$$
$$\forall\, s, s_0 \in \mathbb{C}_{\omega(\mathbf{T})}, s \neq s_0, \qquad (9.1)$$

and $\mathbf{G}_c \in H^\infty(\mathbb{C}_\omega)$ for every $\omega > \omega(\mathbf{T})$. The input-output operator $G_c : L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ is continuous and shift-invariant; moreover, for every $\omega > \omega(\mathbf{T})$, $G_c \in \mathscr{B}(L^2_\omega(\mathbb{R}_+, \mathbb{R}))$ and

$$(\mathscr{L}(G_c v))(s) = \mathbf{G}_c(s)(\mathscr{L}(v))(s), \qquad \forall\, s \in \mathbb{C}_\omega,\ v \in L^2_\omega(\mathbb{R}_+, \mathbb{R}).$$

In the remainder of this chapter, let $s_0 \in \mathbb{C}_{\omega(\mathbf{T})}$ be fixed, but arbitrary. For $x^0 \in X$ and $v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$, let x and w_c denote the state and output functions of Σ, respectively, corresponding to the initial condition $x(0) = x^0 \in X$ and the input function v. Then,

$$x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-s} B v(s)\, ds, \qquad \forall\, t \in \mathbb{R}_+, \qquad (9.2)$$

$$x(t) - (s_0 I - A)^{-1} B v(t) \in \mathrm{dom}(C_\Lambda), \qquad \text{a.a. } t \in \mathbb{R}_+,$$

and

$$\dot{x}(t) = Ax(t) + Bv(t)\,, \qquad x(0) = x^0, \text{ a.a. } t \in \mathbb{R}_+, \tag{9.3a}$$

$$w_c(t) = C_\Lambda(x(t) - (s_0I - A)^{-1}Bv(t)) + \mathbf{G}(s_0)v(t), \text{ a.a. } t \in \mathbb{R}_+. \tag{9.3b}$$

The differential equation (9.3a) has to be interpreted in X_{-1}. Note that (9.3b) yields the following formula for the input-output operator G_c:

$$(G_cv)(t) = C_\Lambda\left[\int_0^t \mathbf{T}_{t-s}Bv(s)\,ds - (s_0I - A)^{-1}Bv(t)\right] + \mathbf{G}_c(s_0)v(t),$$
$$\forall\ v \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}), \text{ a.a. } t \in \mathbb{R}_+. \tag{9.4}$$

In the following, we identify Σ and (9.3) and refer to (9.3) as a well-posed system.

Remark 9.1.1. The class of well-posed linear infinite-dimensional systems is rather general: it includes many distributed parameter systems and all time-delay systems (retarded and neutral) which are of interest in applications. ◇

The above formulas for the output, the input-output operator and the transfer function reduce to a more recognizable form for the subclass of regular systems. The well-posed system (9.3) is called *regular* if

$$\lim_{s\to\infty, s\in\mathbb{R}} \mathbf{G}_c(s) = D,$$

exists and is finite. In this case, $x(t) \in \text{dom}(C_\Lambda)$ for a.a. $t \in \mathbb{R}_+$, the output equation (9.3a) and the formula (9.4) for the input-output operator simplify to

$$w_c(t) = C_\Lambda x(t) + Dv(t), \qquad \text{a.a. } t \in \mathbb{R}_+,$$

and

$$(G_cv)(t) = C_\Lambda \int_0^t \mathbf{T}_{t-s}Bv(s)\,ds + Dv(t), \quad \forall\ v \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}), \text{ a.a. } t \in \mathbb{R}_+,$$

respectively; moreover, $(sI - A)^{-1}B\mathbb{R} \subset \text{dom}(C_\Lambda)$ for all $s \in \text{res}(A)$ and we have

$$\mathbf{G}_c(s) = C_\Lambda(sI - A)^{-1}B + D, \quad \forall\ s \in \mathbb{C}_{\omega(\mathbf{T})}.$$

The number D is called the feedthrough of (9.3).

Definition. The well-posed system (9.3) is called *strongly stable* if the following four conditions are satisfied:

(i) G_c is L^2-stable, that is, $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, \mathbb{R}))$, or, equivalently, $\mathbf{G}_c \in H^\infty(\mathbb{C}_0)$;

(ii) $\mathbf{T}$ is strongly stable, that is, $\lim_{t\to\infty} \mathbf{T}_t z = 0$ for all $z \in X$;

(iii) B is an infinite-time admissible control operator, that is, there exists $\alpha \geq 0$ such that

$$\left\| \int_0^\infty \mathbf{T}_s B v(s)\, ds \right\| \leq \alpha \|v\|_{L^2(\mathbb{R}_+,\mathbb{R})}, \qquad \forall\, v \in L^2(\mathbb{R}_+,\mathbb{R});$$

(iv) C is an infinite-time admissible observation operator, that is, there exists $\beta \geq 0$ such that

$$\left(\int_0^\infty \|C\mathbf{T}_s z\|^2\, ds \right)^{1/2} \leq \beta \|z\|, \qquad \forall\, z \in X_1.$$

The system (9.3) is called *exponentially stable* if $\omega(\mathbf{T}) < 0$. It is clear that exponential stability implies strong stability, however, the converse is not true; for an example of a partial differential equation system which is strongly, but not exponentially stable, see [59].

Finally, for the application of the results in Chapter 8 to well-posed systems, we require several technical results.

Lemma 9.1.2. *Assume the the control operator B or the observation operator C is bounded. Then system* (9.3) *is regular. Moreover, the inverse Laplace transform of the transfer function $\mathbf{G}_c$, or, equivalently, the impulse response of* (9.3), *is in $L^2_\omega(\mathbb{R}_+,\mathbb{R}) + \mathbb{R}\delta_0$ for any $\omega > \omega(\mathbf{T})$, where δ_0 denotes the unit mass at* 0.

A proof of Lemma 9.1.2 can be found in [36] (see [36], Lemma 2.3).

Remark 9.1.3. Note that in particular, if B or C is bounded, it follows from Lemma 9.1.2 that the impulse response of system (9.3) (that is, the convolution kernel of G_c) is a finite (signed) Borel measure on $\mathbb{R}_+$. ◇

Lemma 9.1.4. *Assume that $\mathbf{T}$ is strongly stable, $0 \in \operatorname{res}(A)$ and B is infinite-time admissible. Let $v \in L^2_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R})$ and $v^\infty \in \mathbb{R}$ be such that $v - v^\infty\vartheta_c \in L^2(\mathbb{R}_+,\mathbb{R})$. Then for all $x^0 \in X$, the solution x of* (9.3a) *satisfies*

$$\lim_{t\to\infty} \|x(t) + A^{-1}Bv^\infty\| = 0.$$

If $v^\infty = 0$, then the conclusion remains true even if $0 \notin \operatorname{res}(A)$.

The proof of Lemma 9.1.4 can be found in Appendix 6.

Lemma 9.1.5. *Assume that $\mathbf{T}$ is strongly stable, $0 \in \operatorname{res}(A)$ and B is infinite-time admissible. Let the input $v \in L^2_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R})$ of* (9.3a) *be given by $v = \mathcal{H}w$, where $w \in F(\mathbb{Z}_+,\mathbb{R})$ is such that $\Delta w \in l^2(\mathbb{Z}_+,\mathbb{R})$. Then, for all $x^0 \in X$, the solution x of* (9.3a) *satisfies*

$$\lim_{t\to\infty} \|x(t) + A^{-1}Bv(t)\| = 0.$$

The proof of Lemma 9.1.5 can be found in Appendix 6.

Lemma 9.1.6. *Assume that the well-posed system* (9.3) *is strongly stable and* $0 \in \operatorname{res}(A)$. *Then* $\mathbf{G}_c$ *satisfies assumption* (A_c).

Proof. If (9.3) is strongly stable, then $\mathbf{G}_c \in H^\infty(\mathbb{C}_0)$ and hence $\mathbf{G}_c$ is analytic on $\mathbb{C}_0$. If additionally $0 \in \operatorname{res}(A)$, then $\mathbf{G}_c$ can be analytically extended to a neighbourhood of 0. Hence the evaluation $\mathbf{G}_c(0)$ of $\mathbf{G}_c(s)$ at $s = 0$ is meaningful, and (9.1) holds for $s_0 = 0$ and for $s \in \mathbb{C}_{\omega(\mathbf{T})}$, that is,

$$(\mathbf{G}_c(s) - \mathbf{G}_c(0))/s = C(sI - A)^{-1}A^{-1}B, \qquad \forall\, s \in \mathbb{C}_{\omega(\mathbf{T})}.$$

Hence we see that

$$\limsup_{s \to \infty, s \in \mathbb{C}_0} \left| \frac{1}{s}(\mathbf{G}_c(s) - \mathbf{G}_c(0)) \right| < \infty$$

and assumption (A_c) holds. □

9.2 Sampled-data low-gain integral control of well-posed state-space systems

Throughout this section let $u \in F(\mathbb{Z}_+, \mathbb{R})$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a static non-linearity, let $k, \rho \in \mathbb{R}$, d_c be an external disturbance and consider the well-posed system (9.3), with input non-linearity $v = \varphi \circ u_c$, that is,

$$\dot{x} = Ax + B(\varphi \circ u_c), \quad x(0) = x^0 \in X, \tag{9.5a}$$

$$w_c = C_\Lambda(x - (s_0 I - A)^{-1}B(\varphi \circ u_c)) + \mathbf{G}_c(s_0)(\varphi \circ u_c), \tag{9.5b}$$

controlled by the sampled-data integrator

$$u_c(t) = (\mathcal{H}u)(t), \qquad t \in \mathbb{R}_+, \tag{9.6a}$$

$$(\Delta u)(n) = k(\rho - (\mathcal{S}(d_c + w_c))(n)), \quad u(0) = u^0 \in \mathbb{R},\ n \in \mathbb{Z}_+. \tag{9.6b}$$

From (9.5b), (9.2) and (9.4) we have,

$$\begin{aligned} w_c(t) &= C_\Lambda(x(t) - (s_0 I - A)^{-1}B(\varphi \circ u_c)(t)) + \mathbf{G}_c(s_0)(\varphi \circ u_c)(t) \\ &= C_\Lambda \Big[\mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-s} B(\varphi \circ u_c)(s)\, ds \\ &\qquad\qquad - (s_0 I - A)^{-1}B(\varphi \circ u_c)(t) \Big] + \mathbf{G}_c(s_0)(\varphi \circ u_c)(t) \\ &= C_\Lambda \mathbf{T}_t x^0 + (G_c(\varphi \circ u_c))(t). \end{aligned}$$

Consequently, with $g_c(t) := C_\Lambda \mathbf{T}_t x^0$, it follows from (9.6a) that

$$\mathcal{S}w_c = \mathcal{S}g_c + \mathcal{S}(G_c(\mathcal{H}(\varphi \circ u))). \tag{9.7}$$

Setting $g := \mathcal{S}g_c$, $d := \mathcal{S}d_c$ and $G := \mathcal{S}G_c\mathcal{H}$, it follows from (9.6b) and (9.7) that u satisfies

$$u = u^0\vartheta + kJ(\rho\vartheta - d - (g + G(\varphi \circ u))). \tag{9.8}$$

Theorem 9.2.1. *Assume that the well-posed system (9.3) is exponentially stable and $\mathbf{G}_c(0) > 0$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous and non-decreasing. Suppose that $d_c = d_{c1} + d_{c2}\vartheta_c$ with $d_{c1} \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$ and $d_{c2} \in \mathbb{R}$. Let $\rho \in \mathbb{R}$ and assume that $(\rho - d_{c2})/\mathbf{G}_c(0) \in \operatorname{im}\varphi$. Under these conditions the following statements hold.*

1. *Assume that $\varphi - \varphi(0) \in \mathscr{S}(a)$ for some $a \in (0, \infty)$. Then there exists a constant $k^* \in (0, \infty)$ (depending on G_c, φ and ρ) such that for all $k \in (0, k^*)$, the unique solution (x, u_c) of the feedback system given by (9.5) and (9.6) is defined on $\mathbb{R}_+$, the limits $\lim_{t\to\infty} x(t) =: x^\infty$ (in X) and $\lim_{t\to\infty} u_c(t) =: u^\infty$ exist and satisfy $x^\infty = -((\rho - d_{c2})/\mathbf{G}_c(0))A^{-1}B$ and $\varphi(u^\infty) = (\rho - d_{c2})/\mathbf{G}_c(0)$,*

   $$e_c = \rho - d_c - w_c \in L^2(\mathbb{R}_+, \mathbb{R}) \quad \text{and} \quad \varphi \circ u_c - \varphi(u^\infty)\vartheta_c \in L^2(\mathbb{R}_+, \mathbb{R}).$$

   Moreover,

   $$\lim_{t\to\infty} e_c(t) = 0,$$

   provided that $\mathbf{T}_{t_0}x^0 \in X_1$ for some $t_0 \geq 0$, $\lim_{t\to\infty} d_{c1}(t) = 0$ and G_c satisfies assumption (B_c) (see (7.10)).

2. *Assume that φ is globally Lipschitz continuous with Lipschitz constant $\lambda > 0$. Then the conclusions of statement 1 are valid with $k^* = 1/|\lambda f_J(G)|$ (where $f_J(G)$ is given by (5.2)).*

Proof. Let (x, u_c) be the unique solution of the feedback system given by (9.5) and (9.6). By (9.8), u_c satisfies

$$u_c = u^0\vartheta_c + k\mathcal{H}J(\rho\vartheta - \mathcal{S}(d_c + g_c + G_c(\varphi \circ u_c))) \tag{9.9}$$

where $g_c(t) = C_\Lambda \mathbf{T}_t x^0$. In order to apply Theorem 8.1.1 to (9.9) we need to verify the relevant assumptions. By exponential stability, $G_c \in \mathscr{B}(L^2_\alpha(\mathbb{R}_+, \mathbb{R}))$ for some $\alpha < 0$. It remains to show that $g_c \in L^2_\beta(\mathbb{R}_+, \mathbb{R})$ for some $\beta < 0$. To this end note that if $\beta \in (\omega(\mathbf{T}), 0)$ then, by exponential stability, $g_c \in L^2_\beta(\mathbb{R}_+, \mathbb{R})$.

Proof of Statement 1: An application of Theorem 8.1.1 shows that there exists a constant $k^* \in (0, \infty)$ such that if $k \in (0, k^*)$, $\lim_{t\to\infty} u_c(t) =: u^\infty$ exists and satisfies $\varphi(u^\infty) = (\rho - d_{c2})/\mathbf{G}_c(0)$, $e_c \in L^2(\mathbb{R}_+, \mathbb{R})$ and $\varphi \circ u_c - \varphi(u^\infty)\vartheta_c \in L^2(\mathbb{R}_+, \mathbb{R})$. For the rest of the proof of statement 1, let $k \in (0, k^*)$. If $\mathbf{T}_{t_0}x^0 \in X_1$ for some $t_0 \geq 0$, it follows from the exponential stability of $\mathbf{T}$ that

$$\lim_{t\to\infty} g_c(t) = \lim_{t\to\infty} C\mathbf{T}_{t-t_0}(\mathbf{T}_{t_0}x^0) = 0.$$

Hence, if $\mathbf{T}_{t_0}x^0 \in X_1$ for some $t_0 \geq 0$, $\lim_{t\to\infty} d_{c1}(t) = 0$ and G_c satisfies assumption (B_c), then Theorem 8.1.1 guarantees that $\lim_{t\to\infty} e_c(t) = 0$. By Lemma 9.1.4, $x(t) \to -A^{-1}B\varphi(u^\infty)$ (in X) as $t \to \infty$, showing that $x(t)$ converges to $x^\infty := -((\rho - d_{c2})/\mathbf{G}_c(0))A^{-1}B$ as $t \to \infty$.

Proof of Statement 2: This follows immediately from the application of Theorem 8.1.1. □

We now consider system (9.5) controlled by the sampled-data integrator

$$\begin{aligned} u_c(t) &= (\mathcal{H}u)(t), \qquad t \in \mathbb{R}_+, && (9.10a)\\ \xi(n+1) &= \xi(n) + k(\rho - (\mathcal{S}(d_c + w_c))(n)), \qquad \xi(0) = \xi^0, && (9.10b)\\ u(n) &= \xi(n) + k(\rho - (\mathcal{S}(d_c + w_c))(n)), && (9.10c)\end{aligned}$$

where ξ denotes the integrator state. Again setting $g := \mathcal{S}g_c$, $d := \mathcal{S}d_c$ and $G := \mathcal{S}G_c\mathcal{H}$, it follows from (9.10b), (9.10c) and (9.7) that u satisfies

$$u = \xi^0 + kJ_0(\rho\vartheta - d - (g + G(\varphi \circ u))). \tag{9.11}$$

Theorem 9.2.2. *Assume that the well-posed system* (9.3) *is exponentially stable and* $\mathbf{G}_c(0) > 0$. *Let* $\varphi : \mathbb{R} \to \mathbb{R}$ *be locally Lipschitz continuous and non-decreasing. Suppose that* $d_c = d_{c1} + d_{c2}\vartheta_c$ *with* $d_{c1} \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ *for some* $\alpha < 0$ *and* $d_{c2} \in \mathbb{R}$. *Let* $\rho \in \mathbb{R}$ *and assume that* $(\rho - d_{c2})/\mathbf{G}_c(0) \in \operatorname{im}\varphi$. *Let* (x, u_c) *be a solution of the feedback system given by* (9.5) *and* (9.10). *Under these conditions the following statements hold.*

1. *Assume that* $\varphi - \varphi(0) \in \mathscr{S}(a)$ *for some* $a \in (0, \infty)$. *Then there exists a constant* $k^* \in (0, \infty]$ *(depending on* G_c, φ *and* ρ*) such that for all* $k \in (0, k^*)$, *the limits* $\lim_{t\to\infty} x(t) =: x^\infty$ *(in* X*) and* $\lim_{t\to\infty} u_c(t) =: u^\infty$ *exist and satisfy* $x^\infty = -((\rho - d_{c2})/\mathbf{G}_c(0))A^{-1}B$ *and* $\varphi(u^\infty) = (\rho - d_{c2})/\mathbf{G}_c(0)$,

   $$e_c = \rho - d_c - w_c \in L^2(\mathbb{R}_+, \mathbb{R}) \quad \textit{and} \quad \varphi \circ u_c - \varphi(u^\infty)\vartheta_c \in L^2(\mathbb{R}_+, \mathbb{R}).$$

   Moreover,

   $$\lim_{t\to\infty} e_c(t) = 0,$$

   provided that $\mathbf{T}_{t_0}x^0 \in X_1$ *for some* $t_0 \geq 0$, $\lim_{t\to\infty} d_{c1}(t) = 0$ *and* G_c *satisfies assumption* (B_c). *If* $f_{J_0}(G) = 0$, *then the above conclusions are valid with* $k^* = \infty$ *(where* $f_{J_0}(G)$ *is given by* (5.12)*).*

2. *Assume that φ is globally Lipschitz continuous with Lipschitz constant $\lambda > 0$. Then the conclusions of statement 1 are valid with $k^* = 1/|\lambda f_{J_0}(G)|$, where $1/0 := \infty$.*

3. *Under the assumption $f_{J_0}(G) > 0$, the conclusions of statement 1 are valid with $k^* = \infty$.*

Proof. Let (x, u_c) be a solution of the feedback system given by (9.5) and (9.10). By (9.11), u_c satisfies

$$u_c = \xi^0 \vartheta_c + k\mathcal{H}J_0(\rho\vartheta - \mathcal{S}(d_c + g_c + G_c(\varphi \circ u_c))) \tag{9.12}$$

where $g_c(t) = C_\Lambda \mathbf{T}_t x^0$. In order to apply Theorem 8.1.3 to (9.12) we need to verify the relevant assumptions. Arguments identical to those in the proof of Theorem 9.2.1 show that the relevant assumptions are satisfied. Statements 1 and 2 follow from the application of Theorem 8.1.3 and the arguments used to prove statements 1 and 2 of Theorem 9.2.1. Statement 3 follows immediately from the application of Theorem 8.1.3. □

Let φ, ρ and d_c be as before and let $\kappa : \mathbb{Z}_+ \to \mathbb{R}$ be a time-varying gain. As previously, we consider system (9.5) but now controlled by the sampled-data integrator

$$u_c(t) = (\mathcal{H}u)(t), \qquad t \in \mathbb{R}_+, \tag{9.13a}$$

$$(\Delta u)(n) = \kappa(n)(\rho - (\mathcal{S}(d_c + w_c))(n)), \;\; u(0) = u^0 \in \mathbb{R}, \; n \in \mathbb{Z}_+. \tag{9.13b}$$

Again setting $g := \mathcal{S}g_c$, $d := \mathcal{S}d_c$ and $G := \mathcal{S}G_c\mathcal{H}$, it follows from (9.13b) and (9.7) that u satisfies

$$u = u^0\vartheta + J(\kappa(\rho\vartheta - d - (g + (G(\varphi \circ u))))). \tag{9.14}$$

Theorem 9.2.3. *Assume that the well-posed system* (9.3) *is strongly stable,* $0 \in \operatorname{res}(A)$ *and* $\mathbf{G}_c(0) > 0$. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be non-decreasing and globally Lipschitz continuous with Lipschitz constant $\lambda > 0$, $d_c = d_{c1} + d_{c2}\vartheta_c$ with $d_{c1} \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$ and $d_{c2} \in \mathbb{R}$, $\rho - d_{c2} \in \mathscr{R}(G_c, \varphi)$ and $\kappa : \mathbb{Z}_+ \to \mathbb{R}$ is bounded and non-negative with*

$$\limsup_{n\to\infty} \kappa(n) < 1/|\lambda f_{0,J}(G)|,$$

where $f_{0,J}(G)$ is given by (5.12). *Then, for (x, u_c) the solution of the feedback system given by* (9.5), (9.6a) *and* (9.13b), *and all $x^0 \in X$, the following statements hold.*

1. *The limit $(\varphi \circ u_c)^\infty := \lim_{t\to\infty} \varphi(u_c(t))$ exists and is finite and*

$$\lim_{t\to\infty} \|x(t) + A^{-1}B(\varphi \circ u_c)^\infty\| = 0,$$

and $\Delta(\varphi \circ u) \in l^2(\mathbb{Z}_+, \mathbb{R})$.

2. *If* $\kappa \notin l^1(\mathbb{Z}_+, \mathbb{R})$*, then the error signal* $e_c := \rho\vartheta_c - d_c - w_c$ *can be split into* $e_c = e_{c1} + e_{c2}$*, where* $\lim_{t\to\infty} e_{c1}(t) = 0$ *and* $e_{c2} \in L^2(\mathbb{R}_+, \mathbb{R})$*. Moreover,*

$$\lim_{t\to\infty} e_c(t) = 0,$$

provided that $\mathbf{T}_{t_0}x^0 \in X_1$ *for some* $t_0 \geq 0$*,* $\lim_{t\to\infty} d_{c1}(t) = 0$ *and* G_c *satisfies assumption* (B_c) *(see (7.10)).*

3. *If* $\rho - d_{c2}$ *is an interior point of* $\mathscr{R}(G_c, \varphi)$*, then* u_c *is bounded.*

Proof. Let (x, u_c) be the unique solution of the feedback system given by (9.5) and (9.13). By (9.14), u_c satisfies,

$$u_c = u^0\vartheta_c + \mathcal{H}J(\kappa(\rho\vartheta - \mathcal{S}(d_c + g_c + G_c(\varphi \circ u_c)))) \tag{9.15}$$

where $g_c(t) := C_\Lambda \mathbf{T}_t x^0$. In order to apply Theorem 8.1.8 to (9.15), we need to verify the relevant assumptions. By strong stability and the fact that $0 \in \mathrm{res}(A)$, it is clear from Lemma 9.1.6 that $\mathbf{G}_c$ satisfies assumption (A_c). By the infinite-time admissibility of C we have $g_c \in L^2(\mathbb{R}_+, \mathbb{R})$. We are now in a position to apply Theorem 8.1.8 to (9.15).

Proof of Statement 1: The fact that, $\lim_{t\to\infty} \varphi(u_c(t)) =: (\varphi \circ u_c)^\infty$ exists and is finite follows immediately from the application of Theorem 8.1.8. To see that $\lim_{t\to\infty} \|x(t) + A^{-1}B(\varphi \circ u_c)^\infty\| = 0$ (in X), we apply Lemma 9.1.5 with $w = \varphi \circ u$ (noting that $\Delta w = \Delta(\varphi \circ u) \in l^2(\mathbb{Z}_+, \mathbb{R})$ and $\lim_{t\to\infty} v(t) = \lim_{t\to\infty}(\mathcal{H}w)(t) = \lim_{t\to\infty} \varphi(u_c(t)) =: (\varphi \circ u_c)^\infty$).

Proof of Statement 2: If $\mathbf{T}_{t_0}x^0 \in X_1$ for some $t_0 \geq 0$, then, as in the proof of Theorem 9.2.1, $\lim_{t\to\infty} g_c(t) = 0$. Statement 2 now follows from the application of Theorem 8.1.8.

Proof of Statement 3: This follows immediately from the application of Theorem 8.1.8. □

Let φ, ρ, κ and d_c be as before and let $\psi : \mathbb{R} \to \mathbb{R}$ be a static non-linearity. We now consider system (9.5) with bounded observation operator C (in particular, if C is bounded, $C = C_\Lambda$) . Note that, by Lemma 9.1.2, this means system (9.5) is now regular, that is,

$$\dot{x} = Ax + B(\varphi \circ u_c), \quad x(0) = x^0, \tag{9.16a}$$

$$w_c = Cx + D(\varphi \circ u_c). \tag{9.16b}$$

Boundedness of C implies that the impulse response of G_c is a finite Borel measure on $\mathbb{R}_+$, see Remark 9.1.3. Consequently, we can ideally sample the output $\psi \circ$

$w_c + d_c$. Hence we consider system (9.16), but now controlled by the sampled-data integrator

$$
\begin{aligned}
u_c(t) =& (\mathcal{H}u)(t), \qquad t \in \mathbb{R}_+, && \text{(9.17a)}\\
(\Delta u)(n) =& \kappa(n)(\rho - (\mathcal{S}_I(d_c + \psi \circ w_c))(n)), \ u(0) = u^0 \in \mathbb{R}, \ n \in \mathbb{Z}_+. && \text{(9.17b)}
\end{aligned}
$$

From (9.16b), (9.2) and (9.4) we have,

$$
\begin{aligned}
w_c(t) &= Cx(t) + D(\varphi \circ u_c)(t)\\
&= C\left(\mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-s} B(\varphi \circ u_c)(s)\, ds\right) + D(\varphi \circ u_c)(t)\\
&= C\mathbf{T}_t x^0 + (G_c(\varphi \circ u_c))(t).
\end{aligned}
$$

Consequently, with $g_c(t) := C\mathbf{T}_t x^0$, it follows that

$$
\mathcal{S}_I w_c = \mathcal{S}_I g_c + \mathcal{S}_I(G_c(\mathcal{H}(\varphi \circ u))). \tag{9.18}
$$

Setting $g := \mathcal{S}_I g_c$, $d := \mathcal{S}_I d_c$, noting that by Proposition 8.2.1, ψ and $\mathcal{S}_I$ commute and defining $G := \mathcal{S}_I G_c \mathcal{H}$, it follows from (9.17b) and (9.18) that u satisfies

$$
u = u^0 \vartheta + J(\kappa(\rho\vartheta - d - \psi(g + (G(\varphi \circ u))))). \tag{9.19}
$$

Theorem 9.2.4. *Assume that the well-posed system (9.3) is exponentially stable, that C is bounded (hence system (9.3) is regular, by Lemma 9.1.2) and that $\mathbf{G}_c(0) > 0$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ be non-decreasing and globally Lipschitz continuous with Lipschitz constants $\lambda_1 > 0$ and $\lambda_2 > 0$, $d_c = d_{c1} + d_{c2}\vartheta_c$ with $d_{c1} \in L^2(\mathbb{R}_+, \mathbb{R})$, $\mathcal{S}_I d_{c1} \in l^2(\mathbb{Z}_+, \mathbb{R})$, $n \mapsto \sum_{j=n}^{\infty} |d_{c1}(j\tau)| \in l^2(\mathbb{Z}_+, \mathbb{R})$ and $d_{c2} \in \mathbb{R}$, $\rho - d_{c2} \in \mathscr{R}(G_c, \varphi, \psi)$ and $\kappa : \mathbb{Z}_+ \to \mathbb{R}$ is bounded and non-negative with*

$$
\limsup_{n \to \infty} \kappa(n) < 1/|\lambda_1 \lambda_2 f_{0,J}(G)|,
$$

where $f_{0,J}(G)$ is given by (5.12). Then, for (x, u_c) the solution of the feedback system given by (9.16) and (9.17), and all $x^0 \in X$, the following statements hold.

1. *The limit $(\varphi \circ u_c)^\infty := \lim_{t \to \infty} \varphi(u_c(t))$ exists and is finite and*

   $$\lim_{t \to \infty} \|x(t) + A^{-1}B(\varphi \circ u_c)^\infty\| = 0,$$

   and $\Delta(\varphi \circ u) \in l^2(\mathbb{Z}_+, \mathbb{R})$.

2. *If $\kappa \notin l^1(\mathbb{Z}_+, \mathbb{R})$, then the error signal $e_c := \rho\vartheta_c - d_c - \psi \circ w_c$ can be split into $e_c = e_{c1} + e_{c2}$, where $\lim_{t \to \infty} e_{c1}(t) = 0$ and $e_{c2} \in L^2(\mathbb{R}_+, \mathbb{R})$. Moreover,*

   $$\lim_{t \to \infty} e_c(t) = 0,$$

provided that $\lim_{t\to\infty} d_{c1}(t) = 0$.

3. If $\rho - d_{c2}$ *is an interior point of* $\mathscr{R}(G_c, \varphi, \psi)$*, then* u_c *is bounded.*

Proof. Let (x, u_c) be the unique solution of the feedback system given by (9.16) and (9.17). By (9.19), u_c satisfies,

$$u_c = u^0\vartheta_c + \mathcal{H}J(\kappa(\rho\vartheta - \mathcal{S}_I(d_c + \psi(g_c + G_c(\varphi \circ u_c))))) \tag{9.20}$$

where $g_c(t) := C\mathbf{T}_t x^0$. In order to apply Theorem 8.2.2 to (9.20), we need to verify the relevant assumptions. Since C is bounded the impulse response of G_c is a finite Borel measure on $\mathbb{R}_+$ (see Remark 9.1.3). Using the exponential stability of system (9.3), we see that $\mathbf{G}_c$ is analytic in a neighbourhood of 0 and hence, $\mathbf{G}_c$ satisfies assumption (A_c). It is clear by exponential stability that there exists a $\beta < 0$ such that $g_c \in L^2_\beta(\mathbb{R}_+, \mathbb{R}) \subset L^2(\mathbb{R}_+, \mathbb{R})$. It remains to show that $\mathcal{S}_I g_c \in l^2(\mathbb{Z}_+, \mathbb{R})$. We first note that, by exponential stability of the semigroup $\mathbf{T}$, there exist constants $M, \varepsilon > 0$ such that

$$\|\mathbf{T}_t\| \le Me^{-\varepsilon t}, \quad \forall\, t \in \mathbb{R}_+.$$

Consequently, by boundedness of C,

$$|g_c(t)| \le \widetilde{M}e^{-\varepsilon t}, \quad \forall\, t \in \mathbb{R}_+ \tag{9.21}$$

for some constant $\widetilde{M} \ge 0$, that is, $g_c(t) = O(e^{-\varepsilon t})$ as $t \to \infty$. Hence, $\mathcal{S}_I g_c \in l^2(\mathbb{Z}_+, \mathbb{R})$ as required (see Remarks 8.2.3 (b)). We are now in a position to apply Theorem 8.2.2 to (9.20).

Proof of Statement 1: This follows in the same way as the proof of statement 1 of Theorem 9.2.3, but applying Theorem 8.2.2 instead of Theorem 8.1.8.

Proof of Statement 2: Trivially by (9.21),

$$\lim_{t\to\infty} g_c(t) = 0.$$

The conclusions of statement 2 now follow from the application of Theorem 8.2.2.

Proof of Statement 3: This follows immediately from the application of Theorem 8.2.2. □

Remark 9.2.5. Note that it is also possible to obtain state-space versions of Theorems 8.1.10 and 8.2.4 under similar assumptions to those imposed in Theorems 9.2.3 and 9.2.4. ◇

9.3 Notes and references

The results in this chapter form state-space versions of the input-output results obtained in Chapter 8. Note that, for constant gain with input non-linearity the results from Chapter 8 apply to exponentially stable well-posed state-space systems (see Theorems 9.2.1 and 9.2.2); for time-varying gain with input non-linearity the results from Chapter 8 apply to strongly stable state-space systems with $0 \in \text{res}(A)$ (see Theorem 9.2.3); for time-varying gain with input as well as output non-linearities the results from Chapter 8 apply to exponentially stable systems with bounded observation operator (see Theorem 9.2.4). We remark that the results in this chapter improve on the sampled-data results in [37] and [42] in the sense that, the results in [37] and [42] were obtained for exponentially stable regular systems whereas, Theorems 9.2.1, 9.2.2, 9.2.3 and 9.2.4 apply to the larger class of exponentially stable well-posed systems, in some cases (see above) even to strongly stable well-posed systems with $0 \in \text{res}(A)$. The results in this chapter form the basis for some of [6].

Chapter 10

Discrete-time absolute stability theory and stability of linear multistep methods

An important aspect of numerical analysis is the study of the long term behaviour of solutions of numerical methods. In this chapter we show how special cases of some of the discrete-time absolute theory contained in Chapter 4 can be applied to linear multistep methods. Consequently, we derive results on stability of linear multistep methods.

10.1 An introduction to linear multistep methods

Much of the material in this section is standard. For a more detailed introduction to linear multistep methods see, for example, [1], [26], [27] and [29].

In this chapter we consider the initial-value problem,

$$\frac{dy}{dt}(t) = f(t, y(t)), \quad t \in \mathbb{R}_+, \quad y(0) = y_0, \tag{10.1}$$

defined on a possibly infinite-dimensional Hilbert space U. Here $f : \mathbb{R}_+ \times U \to U$ satisfies suitable regularity conditions (in the least we assume that f is Lipschitz with respect to the norm in U), so that (10.1) has a unique solution. In numerical analysis solutions of (10.1) can be approximated from so called *linear multistep methods* defined below.

Definition. A general *q-step linear multistep method* (LMM) is defined by the equation,

$$\sum_{j=0}^{q} \alpha_j u(n+j) = h \sum_{j=0}^{q} \beta_j f((n+j)h,\, u(n+j)), \qquad n \in \mathbb{Z}_+, \tag{10.2}$$

with fixed time-step $h > 0$, constants $\alpha_j,\ \beta_j \in \mathbb{R}$ $(j = 0, 1, \ldots, q)$ independent of h, n and the underlying differential equation, with $\alpha_q > 0$ and given initial data $u(0), \ldots, u(q-1)$.

We shall often refer to equation (10.2) simply as method (10.2) or, LMM (10.2). When $\beta_q = 0$, LMM (10.2) is said to be *explicit*; otherwise it is *implicit*.

We define the *polynomials* ρ, σ by,

$$\rho(z) := \sum_{j=0}^{q} \alpha_j z^j, \qquad \sigma(z) := \sum_{j=0}^{q} \beta_j z^j.$$

We assume throughout this chapter that ρ and σ are coprime. The LMM (10.2) is completely specified by the two polynomials ρ and σ in the sense that, given polynomials ρ, σ we can determine a LMM of the form (10.2) and conversely given a LMM of the form (10.2) we can define polynomials ρ and σ.

We now give some examples of linear multistep methods.

Example 10.1.1. The backward differentiation formulae (BDFs) are given by,

$$\sigma(z) = z^q, \quad \text{and} \quad \rho(z) = \sum_{k=1}^{q} \frac{1}{k} z^{q-k}(z-1)^k.$$

Given expressions for ρ and σ, we now determine expressions for LMM (10.2) for given values of q.

1. $q = 1$: Then $\sigma(z) = z$ and $\rho(z) = z - 1$. This gives coefficients, $\alpha_0 = -1,\ \alpha_1 = 1,\ \beta_0 = 0,\ \beta_1 = 1$. Hence from (10.2) we obtain the method,

$$u(n+1) - u(n) = hf((n+1)h, u(n+1)).$$

   This is the well-known *Euler method.*

2. $q = 2$: Then $\sigma(z) = z^2$ and $\rho(z) = (3/2)z^2 - 2z + 1/2$. This gives coefficients, $\alpha_0 = 1/2,\ \alpha_1 = -2,\ \alpha_2 = 3/2,\ \beta_0 = 0,\ \beta_1 = 0,\ \beta_2 = 1$. Hence from (10.2) we obtain the method,

$$\frac{3}{2}u(n+2) - 2u(n+1) + \frac{1}{2}u(n) = hf((n+2)h, u(n+2)).$$

◇

We shall now discuss zero-stability of solutions of (10.2) and in so doing, introduce some key definitions.

We begin by considering the case $f \equiv 0$. From (10.2) we now have the linear difference equation,

$$\sum_{j=0}^{q} \alpha_j u(n+j) = 0, \qquad n \in \mathbb{Z}_+. \tag{10.3}$$

Proposition 10.1.2. *Let $z \in \mathbb{C}$. If $\rho(z) = 0$, then $u(n) = z^n$ is a solution of* (10.3). *If, additionally, $\rho'(z) = 0$, then $u(n) = nz^n$ is also a solution of* (10.3).

Proof. Assume that $\rho(z) = 0$ for some $z \in \mathbb{C}$. With $u(n) = z^n$ it follows from (10.3) that

$$\sum_{j=0}^{q} \alpha_j u(n+j) = \sum_{j=0}^{q} \alpha_j z^{n+j} = z^n \left(\sum_{j=0}^{q} \alpha_j z^j \right) = z^n \rho(z) = 0.$$

Hence $u(n) = z^n$ is a solution of (10.3). Suppose further, $\rho'(z) = 0$. With $u(n) = nz^n$ it follows from (10.3) that

$$\begin{aligned} \sum_{j=0}^{q} \alpha_j u(n+j) = \sum_{j=0}^{q} \alpha_j (n+j) z^{n+j} &= z^n \left(n \sum_{j=0}^{q} \alpha_j z^j + \sum_{j=1}^{q} \alpha_j j z^j \right) \\ &= z^n (n\rho(z) + z\rho'(z)) = 0. \end{aligned}$$

Hence $u(n) = nz^n$ is a solution of (10.3). □

Definition. A polynomial $p(z)$ satisfies the *root condition* if

$$p(z) = 0 \text{ implies either } |z| < 1, \text{ or } |z| = 1 \text{ and } p'(z) \neq 0.$$

Proposition 10.1.2 motivates the definition of the following notions of zero-stability of method (10.2).

Definition. The method (10.2) is said to be *zero-stable* if ρ satisfies the root condition. The method (10.2) is *strictly zero-stable* if it is zero-stable and $z = 1$ is the only root of ρ on the complex unit circle.

For a "good" LMM we expect that the values $u(n)$ generated by (10.2) tend to the value of the desired exact solution of (10.1) at time t as $h \to 0$. This gives an intuitive notion of convergence of an LMM which we now make more precise.

We assume that, for $T > 0$, $f : \mathbb{R}_+ \times U \to U$ satisfies the following condition,

$$\|f(t,x) - f(t,z)\| \le L\|x - z\|, \qquad \forall\, t \in [0,T],\ x, z \in U.$$

We introduce the following notation. For $h > 0$, we define $N_h := \{n \in \mathbb{Z}_+ \mid nh \in [0,T]\}$.

Definition. We say that method (10.2) is *convergent* if the following is true for all $y_0 \in U$. Let $\eta_0, \eta_1, \ldots, \eta_{q-1}$ be q functions from $(0, h_0)$ to U where $0 \le h < h_0 = \alpha_q \beta_q^{-1} L^{-1}$ (so that method (10.2) has a unique solution) such that

$$\lim_{h \to 0} \eta_j(h) = y_0, \qquad j = 0, 1, \ldots, q-1,$$

and denote by u the solution of (10.2) having the starting values

$$u(j) = \eta_j(h), \qquad j = 0, 1, \ldots, q-1.$$

Then,

$$\max_{n \in N_h} \|u(n) - y(nh)\| \to 0, \qquad h \to 0,$$

where y denotes the solution of the initial-value problem (10.1). Note that if $\beta_q = 0$, then $h_0 := \infty$.

Convergence of a LMM is an essential property for computation of numerical solutions of initial-value problems of the form (10.1). Without convergence a method is of no practical use.

The condition of zero-stability of method (10.2) has the purpose of preventing a small initial error in the computation of solutions of the initial-value problem (10.1) from growing at such a rate that convergence of (10.2) is no longer guaranteed. However, zero-stability alone does not guarantee convergence. A further condition must be added which ensures that LMM (10.2) is a good approximation to the initial-value problem (10.1).

If LMM (10.2) is to define a "good" method, we expect the difference between the two sides of LMM (10.2) to be small if h is small and if the values $\{u(n)\}_{n \in \mathbb{Z}_+}$ are replaced by $y(nh)$, where y is an exact solution of the initial-value problem (10.1). To this end we now define the concept of consistency of method (10.2).

Definition. Let y be the solution of the initial-value problem (10.1). We say that method (10.2) is *consistent* if,

$$\max_{n \in N_h} \left\| \sum_{j=0}^{q} \alpha_j y((n+j)h) - h \sum_{j=0}^{q} \beta_j f((n+j)h, y((n+j)h)) \right\| = o(h),$$

as $h \to 0$.

The consistency of method (10.2) is equivalent to a purely algebraic condition on the polynomials ρ and σ as seen in the following result. The proof of the following theorem can be found in [27] (see Theorem 3.4 in [27])

Theorem 10.1.3. *The method* (10.2) *is consistent if and only if the polynomials ρ and σ satisfy the conditions*

$$\rho(1) = 0, \qquad \rho'(1) = \sigma(1).$$

Finally we state a necessary and sufficient condition for the convergence of method (10.2) the proof of which can be found in [27] (see Theorem 3.1 in [27]).

Theorem 10.1.4. *The method* (10.2) *is convergent if and only if it is both zero-stable and consistent.*

In order to apply transform methods, we first write (10.2) as a convolution identity. To this end associated with the two polynomials ρ and σ we define sequences $r,\ s \in F(\mathbb{Z}_+, \mathbb{R})$, such that

$$r(n) := \begin{cases} \alpha_{q-n}, & 0 \le n \le q, \\ 0, & n > q, \end{cases} \qquad s(n) := \begin{cases} \beta_{q-n}, & 0 \le n \le q, \\ 0, & n > q. \end{cases} \tag{10.4}$$

We observe that

$$\hat{r}(z) = z^{-q}\rho(z), \qquad \hat{s}(z) = z^{-q}\sigma(z)\,; \quad z \in \mathbb{C}\,, \tag{10.5}$$

In applications, the boundedness of several related quantities is considered: the numerical solution, the numerical error and the difference between two numerical solutions. The following lemma is applicable in all these cases.

Lemma 10.1.5. *Suppose that $D(n) \in U$ for $n \ge q$, and that $\varphi : \mathbb{Z}_+ \times U \to U$. Suppose also that $x : \mathbb{Z}_+ \to U$ satisfies*

$$\sum_{j=0}^{q} \alpha_j x(n+j) = \sum_{j=0}^{q} \beta_j \varphi(n+j,\, x(n+j)) + D(n+q), \quad n \in \mathbb{Z}_+.$$

Then, for $r,\ s \in F(\mathbb{Z}_+, \mathbb{R})$ as defined in (10.4),

$$r * x = s * (\varphi \circ x) + v, \tag{10.6}$$

where v is given by

$$v(n) := \begin{cases} (r * x)(n) - (s * (\varphi \circ x))(n), & 0 \le n \le q-1, \\ D(n), & n \ge q. \end{cases}$$

Proof. Considering the LHS of (10.6), the finite support of r implies that

$$(r * x)(n+q) = \sum_{k=0}^{n+q} r(n+q-k)x(k) = \sum_{j=0}^{q} r(q-j)x(n+j)$$
$$= \sum_{j=0}^{q} \alpha_j x(n+j), \quad n \in \mathbb{Z}_+.$$

Similarly, considering the RHS of (10.6),

$$(s * (\varphi \circ x))(n+q) + v(n+q) = \sum_{k=0}^{n+q} s(n+q-k)(\varphi \circ x)(k) + D(n+q)$$
$$= \sum_{j=0}^{q} s(q-j)(\varphi \circ x)(n+j) = \sum_{j=0}^{q} \beta_j \varphi(n+j,\, x(n+j)) + D(n+q),\ n \in \mathbb{Z}_+.$$

This implies that, for $n \geq q$,

$$(r * x)(n) = (s * (\varphi \circ x))(n) + v(n). \tag{10.7}$$

By the construction of v, (10.7) also holds for $0 \leq n \leq q-1$. Thus, the sequence identity (10.6) is true. □

We recall that ultimately, we are considering the initial-value problem (10.1) where $f : \mathbb{R}_+ \times U \to U$ satisfies suitable regularity conditions. We describe three situations in numerical analysis to which Lemma 10.1.5 applies.

Case 1: The numerical solution. For the method given by (10.2), Lemma 10.1.5 may be applied with

$$x(n) := u(n); \quad \varphi(n,\, \xi) := hf(nh,\, \xi),\ \xi \in U; \quad D(n+q) := 0, \ \ n \in \mathbb{Z}_+.$$

Case 2: The numerical error. Let y be the solution of (10.1). The *truncation error* $T(n+q) \in U$, $n \in \mathbb{Z}_+$, is defined by

$$\sum_{j=0}^{q} \alpha_j y((n+j)h) = h \sum_{j=0}^{q} \beta_j f((n+j)h,\, y((n+j)h)) + hT(n+q), \qquad n \in \mathbb{Z}_+.$$

Lemma 10.1.5 may be applied with

$$\begin{aligned} x(n) &:= y(nh) - u(n), \quad n \in \mathbb{Z}_+, \\ \varphi(n,\, \xi) &:= hf(nh,\, y(nh)) - hf(nh,\, y(nh) - \xi), \quad n \in \mathbb{Z}_+,\ \xi \in U, \\ D(n) &:= hT(n), \quad n \geq q, \end{aligned}$$

where $u : \mathbb{Z}_+ \to U$ is the solution of (10.2).

Case 3: Difference of two numerical solutions. If $u_1 : \mathbb{Z}_+ \to U$, $u_2 : \mathbb{Z}_+ \to U$, are two solutions of (10.2), then Lemma 10.1.5 may be applied with

$$\begin{aligned} x(n) &:= u_1(n) - u_2(n), \quad n \in \mathbb{Z}_+, \\ \varphi(n, \xi) &:= hf(nh, u_1(n)) - hf(nh, u_1(n) - \xi), \quad n \in \mathbb{Z}_+, \ \xi \in U, \\ D(n) &:= 0, \quad n \in \mathbb{Z}_+. \end{aligned}$$

10.2 Control theoretic absolute stability results

We consider an absolute stability problem for the feedback system shown in Figure 10.1. The convolution kernel $g : \mathbb{Z}_+ \to \mathbb{C}$, the time-dependent non-linearity $\varphi : \mathbb{Z}_+ \times U \to U$ and the forcing function $w : \mathbb{Z}_+ \to U$ are given.

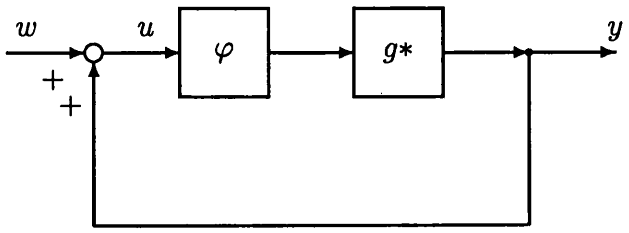

Figure 10.1: Feedback system with non-linearity

From Figure 10.1 we can derive the following governing equations

$$u = y + w, \qquad y = g * (\varphi \circ u),$$

or, equivalently,

$$u = g * (\varphi \circ u) + w. \tag{10.8}$$

The latter equation can be written as the nonlinear discrete-time Volterra equation,

$$u(n) = \sum_{j=0}^{n} g(n-j)\varphi(j, u(j)) + w(n).$$

A solution of (10.8) is a U-valued function u defined on $\mathbb{Z}_+$ satisfying (10.8). Trivially, there exists at least one solution (a unique solution, respectively) of (10.8) if, for every $n \in \mathbb{Z}_+$, the map

$$U \to U, \qquad \xi \mapsto \xi - g(0)\varphi(n, \xi)$$

is surjective (bijective, respectively).

In the following result, part (A) is a special case of Theorem 4.3.4 and part (B) is a special case of Theorem 4.3.6.

Theorem 10.2.1. *Let $g = g_0\vartheta + g_1$, where $g_0 \in (0,\infty)$ and $g_1 \in l^1(\mathbb{Z}_+)$, let φ be sector-bounded in the sense that there exists $a \in (0,\infty]$ such that*

$$\operatorname{Re}\langle\varphi(n,\xi),\xi\rangle \leq -\|\varphi(n,\xi)\|^2/a\,, \quad \forall\,(n,\xi) \in \mathbb{Z}_+ \times U \tag{10.9}$$

and assume that there exists $\varepsilon \geq 0$ such that

$$1/a + \operatorname{Re}\hat{g}(e^{i\theta}) \geq \varepsilon\,, \quad \forall\,\theta \in (0, 2\pi)\,. \tag{10.10}$$

(A) *If $\varepsilon > 0$ and $w \in m^2(\mathbb{Z}_+, U)$, then every solution u of (10.8) has the following properties.*

(A1) *There exists a constant $K > 0$ (depending only on ε, a and g, but not on w) such that*

$$\|u\|_{l^\infty} + \|\triangle_0 u\|_{l^2} + \|\varphi\circ u\|_{l^2} + (\|\operatorname{Re}\langle\varphi\circ u, u\rangle\|_{l^1})^{1/2} \\ + \|J_0(\varphi\circ u)\|_{l^\infty} \leq K\|w\|_{m^2}.$$

(A2) *The limit $\lim_{n\to\infty}\|u(n)\|$ exists and is finite; in particular, if $\dim U = 1$, then $\lim_{n\to\infty} u(n)$ exists.*

(A3) *Under the additional assumptions*

(A3.1) *φ does not depend on time,*

(A3.2) *$\varphi^{-1}(0)\cap B$ is precompact for every bounded set $B \subset U$,*

(A3.3) *$\inf_{\xi\in B}\|\varphi(\xi)\| > 0$ for every bounded closed set $B \subset U$ such that $\varphi^{-1}(0)\cap B = \emptyset$,*

we have that $\lim_{n\to\infty}\operatorname{dist}(u(n),\varphi^{-1}(0)) = 0$.

(A4) *If (A3.1)-(A3.3) and the additional assumption,*

(A4.1) *$\operatorname{cl}(\varphi^{-1}(0))\cap S$ is totally disconnected for every sphere $S \subset U$ centred at 0,*

hold, then $u(n)$ converges as $n\to\infty$.

(B) *If $\varepsilon = 0$ and $w \in m^1(\mathbb{Z}_+, U)$, then every solution u of (10.8) has the following properties.*

(B1) *There exists a constant* $K > 0$ *(depending only on* a *and* g*, but not on* w*) such that*

$$\|u\|_{l^\infty} + (\|\mathrm{Re}\,\langle \varphi \circ u, u + \frac{1}{a}(\varphi \circ u)\rangle\|_{l^1})^{1/2} + \|J_0(\varphi \circ u)\|_{l^\infty} \leq K\|w\|_{m^1}.$$

(B2) *Under the assumptions* (A3.1), (A3.2) *and the additional assumption*

(B2.1) $\sup_{\xi \in B} \mathrm{Re}\,\langle \varphi(\xi), \xi + \varphi(\xi)/a\rangle < 0$ *for every bounded closed set* $B \subset U$ *such that* $\varphi^{-1}(0) \cap B = \emptyset$,

we have that $\lim_{n\to\infty} \mathrm{dist}(u(n), \varphi^{-1}(0)) = 0$.

(B3) *If* (A3.1), (A3.2), (B2.1) *and the additional assumption,*

(B3.1) φ *is continuous,*

hold, then $\lim_{n\to\infty}(\Delta_0 u)(n) = 0$. *If further,*

(B3.2) $\varphi^{-1}(0)$ *is totally disconnected,*

then $\lim_{n\to\infty} u(n) =: u^\infty$ *exists with* $u^\infty \in \varphi^{-1}(0)$.

A complete proof of Theorem 10.2.1 (which does not refer to Theorems 4.3.4 and 4.3.6) can be found in [4] (see, Theorem 5.1 in [4]).

Next, we derive a version of Theorem 10.2.1 which yields stability properties of the difference of two solutions of (10.8). In this context, the following incremental sector condition

$$\mathrm{Re}\,\langle \varphi(n,\xi_1) - \varphi(n,\xi_2), \xi_1 - \xi_2\rangle \leq -\|\varphi(n,\xi_1) - \varphi(n,\xi_2)\|^2/a\,, \qquad n \in \mathbb{Z}_+,\ \xi_1, \xi_2 \in U \tag{10.11}$$

is relevant. Let w_1, w_2 be forcing functions and let u_1 and u_2 be corresponding solutions of (10.8), that is, $u_i = g * (\varphi \circ u_i) + w_i$ for $i = 1, 2$. The difference $u_1 - u_2$ satisfies the Volterra equation

$$u_1 - u_2 = g * (\psi \circ (u_1 - u_2)) + w_1 - w_2\,.$$

In the following result, part (A) is a special case of Corollary 4.4.1 and part (B) is a special case of Corollary 4.4.2.

Corollary 10.2.2. *Let* $g = g_0\theta + g_1$*, where* $g_0 \in (0, \infty)$ *and* $g_1 \in l^1$*, let* φ *be incrementally sector-bounded in the sense that* (10.11) *holds for some* $a \in (0, \infty]$ *and assume that there exists* $\varepsilon \geq 0$ *such that* (10.10) *is satisfied. Let* u_1 *and* u_2 *be solutions of* (10.8) *corresponding to forcing functions* w_1 *and* w_2*, respectively.*

(A) *If* $\varepsilon > 0$ *and* $w_1 - w_2 \in m^2(\mathbb{Z}_+, U)$*, then the following statements hold:*

(A1) *There exists a constant $K > 0$ (depending only on ε, a and g, but not on w_1 and w_2) such that*

$$\|u_1 - u_2\|_{l^\infty} + \|\Delta_0(u_1 - u_2)\|_{l^2} + (\|\mathrm{Re}\,\langle \varphi \circ u_1 - \varphi \circ u_2, u_1 - u_2\rangle\|_{l^1})^{1/2} \\ + \|\varphi \circ u_1 - \varphi \circ u_2\|_{l^2} + \|J_0(\varphi \circ u_1 - \varphi \circ u_2)\|_{l^\infty} \leq K\|w_1 - w_2\|_{m^2}\,.$$

(A2) *The limit $\lim_{n\to\infty} \|u_1(n) - u_2(n)\|$ exists and is finite; in particular, if $\dim U = 1$, then $\lim_{n\to\infty}(u_1(n) - u_2(n))$ exists.*

(B) *If $\varepsilon = 0$ and $w_1 - w_2 \in m^1(\mathbb{Z}_+, U)$, then there exists a constant $K > 0$ (depending only on a and g, but not on w_1 and w_2) such that*

$$\|u_1 - u_2\|_{l^\infty} + (\|\mathrm{Re}\,\langle \varphi \circ u_1 - \varphi \circ u_2, u_1 - u_2 + \frac{1}{a}(\varphi \circ u_1 - \varphi \circ u_2)\rangle\|_{l^1})^{1/2} \\ + \|J_0(\varphi \circ u_1 - \varphi \circ u_2)\|_{l^\infty} \leq K\|w_1 - w_2\|_{m^1}\,.$$

10.3 An application of Section 10.2 to linear multistep stability

In this section we apply Theorem 10.2.1 and Corollary 10.2.2 to derive results on the asymptotic behaviour of the solutions of (10.2).

We begin by defining the notion of a positive rational function.

Definition. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be a *rational function* if it can be expressed as the quotient of two polynomials with possibly complex coefficients. We further say that a rational function is *proper* if $\lim_{|z|\to\infty} |f(z)| < \infty$ for all $z \in \mathbb{C}$. A rational function f (with possibly complex coefficients) is called *positive* if

$$\mathrm{Re}\, f(z) \geq 0\,, \quad \text{for all } z \in \mathbb{C} \text{ with } |z| > 1 \text{ and such that } z \text{ is not a pole of } f.$$

The following lemma lists some simple (and well-known) properties of positive rational functions which will be required throughout this section, we include a proof for completeness.

Lemma 10.3.1. *Let f be a rational function such that $f(z) \not\equiv 0$. If f is positive, then the following statements hold:*

(a) *If $z \in \mathbb{C}$ is a zero of f, then $|z| \leq 1$.*

(b) *If $z \in \mathbb{C}$ is a pole of f, then $|z| \leq 1$.*

(c) $\lim_{|z|\to\infty} f(z)$ *exists, is finite and not equal to zero.*

(d) *If* $e^{i\nu}$ *is a zero of* f *for some* $\nu \in [0, 2\pi)$, *then it is simple (that is* $f'(e^{i\nu}) \neq 0$*).*

(e) *If* $e^{i\nu}$ *is a pole of* f *for some* $\nu \in [0, 2\pi)$, *then it is simple (that is* $(1/f)'(e^{i\nu}) \neq 0$*) and* $e^{-i\nu}\mathrm{Res}\,(f, e^{i\nu}) > 0$, *where* $\mathrm{Res}\,(f, e^{i\nu})$ *denotes the residue of* f *at* $z = e^{i\nu}$.

Note that part (c) says that a non-trivial positive rational function has no zeros or poles at ∞.

Proof of Lemma 10.3.1. (a) Let $|z_0| > 1$ and assume that $f(z_0) = 0$. Then

$$f(z) = (z - z_0)^m g(z),$$

where $m \in \mathbb{N}$ and g is a rational function which is holomorphic at $z = z_0$ and $g(z_0) \neq 0$. Let $\rho > 0$ be sufficiently small such that $|z_0 + \rho e^{i\theta}| > 1$ for all $\theta \in [0, 2\pi)$. Then

$$f(z_0 + \rho e^{i\theta}) = g(z_0 + \rho e^{i\theta})\rho^m e^{im\theta}.$$

Hence,

$$\begin{aligned} 0 &\leq \rho^{-m}\mathrm{Re}\, f(z_0 + \rho e^{i\theta}) \\ &= \cos(m\theta)\mathrm{Re}\,(g(z_0 + \rho e^{i\theta})) - \sin(m\theta)\mathrm{Im}\,(g(z_0 + \rho e^{i\theta})). \end{aligned} \tag{10.12}$$

Since $g(z_0 + \rho e^{i\theta})) \to g(z_0) \neq 0$ as $\rho \to 0$ (uniformly in θ), it follows that there exists $\rho > 0$ and $\theta \in [0, 2\pi)$ such that the RHS of (10.12) is negative, contradicting the fact that $\mathrm{Re}\, f(z_0 + \rho e^{i\theta}) \geq 0$.
(b) If $z \in \mathbb{C}$ is a pole of f, then z is a zero of the positive rational function $1/f$. Hence, by (a), it follows that $|z| \leq 1$.
(c) By (a) and (b) all poles and zeros of f are in $\overline{\mathbb{B}}$. Define a rational function

$$g(z) := f(1/z).$$

Then since f is positive, $\mathrm{Re}\, g(z) \geq 0$ for all $z \in \mathbb{B}$ such that z is not a pole of g. It suffices to show that $\lim_{z\to 0} g(z)$ exists, is finite and not equal to zero. Let

$$g(z) = z^m h(z),$$

where $m \in \mathbb{Z}$ and h is a rational function which is holomorphic at $z = 0$ and $h(0) \neq 0$. Let $\rho < 1$. Then, for all $\theta \in [0, 2\pi)$,

$$g(\rho e^{i\theta}) = h(\rho e^{i\theta})\rho^m e^{im\theta}.$$

Hence,

$$0 \le \rho^{-m} \mathrm{Re}\, g(\rho e^{i\theta}) = \cos(m\theta) \mathrm{Re}\,(h(\rho e^{i\theta})) - \sin(m\theta) \mathrm{Im}\,(h(\rho e^{i\theta})). \tag{10.13}$$

Since $h(\rho e^{i\theta})) \to h(0) \neq 0$ as $\rho \to 0$ (uniformly in θ), it follows that there exists $\rho < 1$ and $\theta \in [0, 2\pi)$ such that the RHS of (10.13) is negative, contradicting the fact that $\mathrm{Re}\, g(\rho e^{i\theta}) \ge 0$.
(d) Assume that $e^{i\nu}$ is a zero of f. Then

$$f(z) = (z - e^{i\nu})^m g(z), \tag{10.14}$$

where $m \in \mathbb{N}$ and g is a rational function which is holomorphic at $z = e^{i\nu}$ and $g(e^{i\nu}) \neq 0$. For $\varepsilon > 0$ and $\theta \in [-\pi/2, \pi/2]$, we define

$$z_{\varepsilon,\theta} := e^{i\nu}(1 + \varepsilon e^{i\theta}).$$

It is clear that $|z_{\varepsilon,\theta}| > 1$ and so, for $\varepsilon > 0$ and $\theta \in [-\pi/2, \pi/2]$,

$$\begin{aligned} 0 &\le \varepsilon^{-m} \mathrm{Re}\, f(z_{\varepsilon,\theta}) \\ &= \cos(m\theta) \mathrm{Re}\,(e^{im\nu} g(z_{\varepsilon,\theta})) - \sin(m\theta) \mathrm{Im}\,(e^{im\nu} g(z_{\varepsilon,\theta})). \end{aligned} \tag{10.15}$$

Since $g(z_{\varepsilon,\theta}) \to g(e^{i\nu}) \neq 0$ as $\varepsilon \to 0$ (uniformly in θ), it follows from (10.15) that $m = 1$ (because otherwise there would exist $\varepsilon > 0$ and $\theta \in [-\pi/2, \pi/2]$ such that the RHS of (10.15) is negative). Consequently, the zero at $z = e^{i\nu}$ is simple.
(e) Assume that $e^{i\nu}$ is a pole of f. Replacing m by $-m$ in (10.14), an argument identical to that in the proof of (d) shows that the pole at $z = e^{i\nu}$ is simple. Considering (10.15) when $m = -1$, choosing θ to be $-\pi/2$, 0 and $\pi/2$ and letting $\varepsilon \to 0$, shows that $\mathrm{Im}\,(e^{-i\nu} g(e^{i\nu})) = 0$ and $\mathrm{Re}\,(e^{-i\nu} g(e^{i\nu})) \ge 0$. Since $g(e^{i\nu}) \neq 0$, we conclude that $\mathrm{Re}\,(e^{-i\nu} g(e^{i\nu})) > 0$. The claim now follows from the fact that $\mathrm{Res}\,(f, e^{i\nu}) = g(e^{i\nu})$. □

Definition. A sequence $a \in F(\mathbb{Z}_+)$ is said to be *exponentially decaying* if there exists $\eta \in (0, 1)$ and $M > 0$ such that

$$|a(n)| \le M\eta^n, \quad n \in \mathbb{Z}_+.$$

Note that $a \in F(\mathbb{Z}_+)$ is exponentially decaying if and only if $r_a < 1$ (where r_a is given by (3.5)).

In the following, if $a \in F(\mathbb{Z}_+)$ and $k \in \mathbb{N}$, then

$$a^k := \underbrace{a * a * \cdots * a}_{k \text{ factors}}.$$

For $\xi \in \mathbb{C}$, define

$$\psi_\xi := (0, 1, \xi, \xi^2, \xi^3, \ldots).$$

Note that $\hat{\psi}_\xi(z) = 1/(z - \xi)$.

Lemma 10.3.2. *Let A be a proper rational function. Then there exists a $\mathscr{Z}$-transformable sequence $a \in F(\mathbb{Z}_+)$ such that $a = \mathscr{Z}^{-1}(A)$ and a is of the form*

$$a = \gamma\delta + \sum_{k=1}^{m} \sum_{j=1}^{m_k} \gamma_{kj} \psi_{z_k}^{j}, \tag{10.16}$$

where γ, $\gamma_{kj} \in \mathbb{C}$ are suitable coefficients, the z_k are the poles of A and m_k denotes the multiplicity of z_k.

Proof. If p_k is the principal part of the Laurent expansion of A at z_k, then

$$p_k(z) = \sum_{j=1}^{m_k} \frac{\gamma_{kj}}{(z - z_k)^j},$$

where the γ_{kj} are suitable constants. The function $B := A - \sum_{k=1}^{m} p_k$ is a rational function without any poles, and hence B must be a polynomial. Since A is proper, it follows that B is a constant polynomial equal to some $\gamma \in \mathbb{C}$. Therefore, $A = \gamma + \sum_{k=1}^{m} p_k$, and thus $a := \mathscr{Z}^{-1}(A)$ is of the form (10.16). □

The following corollary is an immediate consequence of Lemma 10.3.2.

Corollary 10.3.3. *For a proper rational function A, the following statements hold:*

(a) *If A is holomorphic in $\overline{\mathbb{E}}_1$, then $\mathscr{Z}^{-1}(A)$ is exponentially decaying; in particular, $\mathscr{Z}^{-1}(A) \in l^1(\mathbb{Z}_+)$.*

(b) *If A is holomorphic in $\mathbb{E}_1$ and has only simple poles $\{z_k\}_{k=1}^{m}$ on the complex unit circle, then*

$$\mathscr{Z}^{-1}(A) = a_0 + \sum_{k=1}^{m} \gamma_k \psi_{z_k},$$

where $a_0 \in F(\mathbb{Z}_+)$ is exponentially decaying and γ_k denotes the residue of A at z_k.

We shall require the following key definition.

Definition. The *linear stability domain* $\mathbb{S}$ for method (10.2) is the set

$$\mathbb{S} := \{\zeta \in \mathbb{C} : \rho(z) - \zeta\sigma(z) \text{ satisfies the root condition}\}.$$

To apply Theorem 10.2.1 and Corollary 10.2.2 to method (10.2), we first require some preliminary results.

Lemma 10.3.4. *For $c \in \mathbb{C}$ and $R > 0$, the following equivalences hold for method* (10.2):

$$\begin{aligned}\mathbb{B}(c,\, R) \subset \mathbb{S} &\iff \inf_{|z|\geq 1} |\rho(z)/\sigma(z) - c| \geq R \\ &\iff \sup_{|z|\geq 1} |\sigma(z)/(\rho(z) - c\sigma(z))| \leq 1/R.\end{aligned}$$

Proof. Assuming $\mathbb{B}(c,\, R) \subset \mathbb{S}$, we deduce that $\mathbb{B}(c,\, R) \subset \operatorname{int}(\mathbb{S})$. Hence, if $\zeta \in \mathbb{B}(c, R)$, then $\rho(z) - \zeta\sigma(z) = 0$ implies that $|z| < 1$. Thus,

$$\inf_{|z|\geq 1} |\rho(z)/\sigma(z) - c| \geq R, \tag{10.17}$$

or, equivalently,

$$\sup_{|z|\geq 1} |\sigma(z)/(\rho(z) - c\sigma(z))| \leq 1/R. \tag{10.18}$$

Conversely, assume (10.18) or, equivalently, (10.17) holds. For $\zeta \in \mathbb{C} \setminus \mathbb{S}$, there exists $z_0 \in \mathbb{C}$ with $|z_0| \geq 1$ and such that $\rho(z_0) - \zeta\sigma(z_0) = 0$. Thus, by (10.17), $|\zeta - c| \geq R$, which implies $\zeta \in \mathbb{C} \setminus \mathbb{B}(c,\, R)$. We deduce that $\mathbb{B}(c,\, R) \subset \mathbb{S}$. □

The following result shows that the inclusion of a disc of the form $\mathbb{B}(-c, c)$ (where $c > 0$) in the linear stability domain of (10.2) is equivalent to the positivity of the rational function $1/(2c) + \sigma/\rho$.

Lemma 10.3.5. *Let $\mathbb{S}$ denote the linear stability domain of* (10.2) *and let $c > 0$. Then*

$$\mathbb{B}(-c, c) \subset \mathbb{S} \quad \iff \quad 1/(2c) + \sigma/\rho \text{ is positive}.$$

Moreover, if $\mathbb{B}(-c, c) \subset \mathbb{S}$, then the following statements hold:

(a) *Method* (10.2) *is zero-stable.*

(b) *If $\rho(e^{i\nu}) = 0$ for some $\nu \in [0, 2\pi)$, then $e^{-i\nu}\sigma(e^{i\nu})/\rho'(e^{i\nu}) > 0$.*

Proof. A straightforward calculation shows that, for all $z \in \mathbb{C}$ such that $\rho(z) \neq 0$, the following equivalences hold

$$\begin{aligned}|\rho(z)/\sigma(z) + c| \geq c &\Leftrightarrow (1/c^2)|\rho(z)|^2\, |1 + c\sigma(z)/\rho(z)|^2 \geq |\sigma(z)|^2 \\ &\Leftrightarrow 1/(2c) + \operatorname{Re}(\sigma(z)/\rho(z)) \geq 0\,.\end{aligned}$$

Now, by Lemma 10.3.4, $\mathbb{B}(-c, c) \subset \mathbb{S}$ if and only if $\inf_{|z|\geq 1} |\rho(z)/\sigma(z) + c| \geq c$, showing that the inclusion $\mathbb{B}(-c, c) \subset \mathbb{S}$ is equivalent to the positivity of

$1/(2c)+\sigma/\rho$. Statements (a) and (b) are now immediate consequences of Lemma 10.3.1. $\square$

The following proposition shows that if $\mathbb{B}(-c,c) \subset \mathbb{S}$, then $\overline{\mathbb{B}}(-c,c) \subset \mathbb{S}$.

Proposition 10.3.6. *Let $\mathbb{S}$ denote the linear stability domain of* (10.2) *and let $c > 0$. If $\mathbb{B}(-c,c) \subset \mathbb{S}$, then $\overline{\mathbb{B}}(-c,c) \subset \mathbb{S}$.*

Proof. Let ζ be on the boundary of $\mathbb{B}(-c,c)$, that is $|\zeta + c| = c$. We have to show that $\zeta \in \mathbb{S}$. To this end let $z_0 \in \mathbb{C}$ be such that

$$\rho(z_0) - \zeta\sigma(z_0) = 0\,. \tag{10.19}$$

Since $\mathbb{B}(-c,c) \subset \mathbb{S}$, it is clear that $|z_0| \leq 1$. Therefore, it is sufficient to prove that if $|z_0| = 1$, then

$$\rho'(z_0) - \zeta\sigma'(z_0) \neq 0\,. \tag{10.20}$$

So let us assume that $|z_0| = 1$. If $\zeta = 0$, then $\rho(z_0) = 0$. By part (b) of Lemma 10.3.5, $\rho'(z_0) - \zeta\sigma'(z_0) = \rho'(z_0) \neq 0$, and so (10.20) holds. Hence, w.l.o.g. we may assume that $\zeta \neq 0$. Using that $|\zeta + c| = c$,

$$\operatorname{Re}\left(\frac{1}{\zeta}\right) = \frac{\operatorname{Re}\zeta}{|\zeta|^2} = \frac{\alpha}{\alpha^2+\beta^2} \tag{10.21}$$

where $\alpha := \operatorname{Re}\zeta$ and $\beta := \operatorname{Im}\beta$. Then,

$$(\alpha + c)^2 + \beta^2 = |\zeta + c|^2 = c^2 \quad \Rightarrow \quad \alpha^2 + \beta^2 = -2c\alpha$$

and so we see from (10.21)

$$\operatorname{Re}(1/\zeta) = \alpha/(-2c\alpha) = -1/(2c)\,.$$

Invoking part (a) of Lemma 10.3.5, we see that the rational function $A := \sigma/\rho - 1/\zeta$ is positive. It follows from (10.19) that $A(z_0) = 0$. By part (d) of Lemma 10.3.1, $A'(z_0) \neq 0$ and so

$$\begin{aligned} 0 \neq A'(z_0) &= \frac{(\rho(z_0) - \zeta\sigma(z_0))\sigma'(z_0) - (\rho'(z_0) - \zeta\sigma'(z_0))\sigma(z_0)}{\rho^2(z_0)} \\ &= -\frac{(\rho'(z_0) - \zeta\sigma'(z_0))\sigma(z_0)}{\rho^2(z_0)}\,, \end{aligned}$$

showing that (10.20) holds. $\square$

We are now in the position to formulate the main result of this section. In order to do this we first define the function $f_h : \mathbb{Z}_+ \times U \to U$ by

$$f_h(n,\xi) := hf(nh,\xi)\,, \qquad (n,\xi) \in \mathbb{Z}_+ \times U.$$

Theorem 10.3.7. *Assume that method* (10.2) *satisfies the following two conditions: $\rho(1) = 0$ and $\rho(e^{i\theta}) \neq 0$ for all $\theta \in (0, 2\pi)$.*

(A) *Assume that there exists $0 < c < \infty$ such that*

$$\|hf(nh, \xi) + c\xi\| \leq c\|\xi\|, \quad (n, \xi) \in \mathbb{Z}_+ \times U, \tag{10.22}$$

and further that $\mathbb{B}(-c_0, c_0) \subset \mathbb{S}$ for some $c_0 > c$. Under these conditions, for every solution $u : \mathbb{Z}_+ \to U$ of (10.2)*, the following statements hold:*

(A1) *There exists a constant $K > 0$ (depending only on c_0, c and (ρ, σ)) such that,*

$$\|u\|_{l^\infty} + \|\triangle_0 u\|_{l^2} + \|f_h \circ u\|_{l^2} + (\|\mathrm{Re}\,\langle f_h \circ u, u\rangle\|_{l^1})^{1/2} + \|J_0(f_h \circ u)\|_{l^\infty} \leq K \left(\sum_{k=0}^{q-1} \|u(k)\|^2\right)^{1/2} .$$

(A2) *The limit $\lim_{n\to\infty} \|u(n)\|$ exists and is finite (in particular, if $\dim U = 1$, then $\lim_{n\to\infty} u(n)$ exists).*

(A3) *Under the additional assumptions*

- (A3.1) *f does not depend on time,*
- (A3.2) *$f^{-1}(0) \cap B$ is precompact for every bounded set $B \subset U$,*
- (A3.3) *$\inf_{\xi \in B} \|f(\xi)\| > 0$ for every bounded closed set $B \subset U$ such that $f^{-1}(0) \cap B = \emptyset$,*

we have that $\lim_{n\to\infty} \mathrm{dist}(u(n), f^{-1}(0)) = 0$.

(A4) *If* (A3.1)-(A3.3) *and*

- (A4.1) *$\mathrm{cl}\,(f^{-1}(0)) \cap S$ is totally disconnected for every sphere $S \subset U$ centred at 0,*

hold, then $u(n)$ converges as $n \to \infty$.

(B) *Assume that there exists $0 < c < \infty$ such that* (10.22) *holds and further that $\mathbb{B}(-c, c) \subset \mathbb{S}$. Under these conditions, for every solution $u : \mathbb{Z}_+ \to U$ of* (10.2)*, the following statements hold:*

(B1) *There exists a constant $K > 0$ (depending only on c and (ρ, σ)) such that,*

$$\|u\|_{l^\infty} + (\|\mathrm{Re}\,\langle f_h \circ u, u + \frac{1}{2c}(f_h \circ u)\rangle\|_{l^1})^{1/2} + \|J_0(f_h \circ u)\|_{l^\infty} \leq K \sum_{k=0}^{q-1} \|u(k)\| .$$

(B2) *Under the assumptions* (A3.1), (A3.2) *and the additional assumption*

(B2.1) $\sup_{\xi\in B} \operatorname{Re}\langle f(\xi), \xi + f(\xi)/(2c)\rangle < 0$ *for every bounded closed set* $B \subset U$ *such that* $\varphi^{-1}(0) \cap B = \emptyset$,

we have that $\lim_{n\to\infty} \operatorname{dist}(u(n), f^{-1}(0)) = 0$.

(B3) *If assumptions* (A3.1), (A3.2), (B2.1) *and the additional assumption,*

(B3.1) f *is continuous,*

hold, then $\lim_{n\to\infty}(\Delta_0 u)(n) = 0$. *If further,*

(B3.2) $f^{-1}(0)$ *is totally disconnected,*

then $\lim_{n\to\infty} u(n) =: u^\infty$ *exists with* $u^\infty \in f^{-1}(0)$.

(C) *Assume that*

$$\operatorname{Re}\langle f(nh,\xi), \xi\rangle \le 0\,, \quad (n,\xi) \in \mathbb{Z}_+ \times U\,. \tag{10.23}$$

If $\{z \in \mathbb{C} : \operatorname{Re} z < 0\} \subset \mathbb{S}$, *then there exists a constant* $K > 0$ *(depending only on* (ρ,σ)*) such that, for every solution* $u : \mathbb{Z}_+ \to U$ *of* (10.2),

$$\|u\|_{l^\infty} + (\|\operatorname{Re}\langle f_h \circ u, u\rangle\|_{l^1})^{1/2} + \|J_0(f_h \circ u)\|_{l^\infty} \le K \sum_{k=0}^{q-1} (\|u(k)\| + \|hf(kh, u(k))\|)\,,$$

and the conclusions of statements (B2) *and* (B3) *hold.*

Remark 10.3.8. (a) In the scalar case (that is, $U = \mathbb{C}$) the inequality (10.22) is equivalent to the condition that $hf(nh,\xi)/\xi \in \mathbb{B}(-c,c)$ for all $(n,\xi) \in \mathbb{Z}_+ \times \mathbb{C}$ with $\xi \ne 0$.

(b) Assume that there exist $c_1, h_1 > 0$ such that

$$\|h_1 f(t,\xi) + c_1\xi\| \le c_1\|\xi\|\,, \quad (t,\xi) \in \mathbb{R}_+ \times U \tag{10.24}$$

and $\mathbb{B}(-c_1, c_1) \subset \mathbb{S}$, so that the conclusions of part (B) hold for $c = c_1$ and $h = h_1$. Letting $h_2 \in (0, h_1)$ and setting $c_2 := c_1 h_2/h_1 < c_1$, it follows trivially from (10.24) that

$$\|h_2 f(t,\xi) + c_2\xi\| \le c_2\|\xi\|\,, \quad (t,\xi) \in \mathbb{R}_+ \times U\,,$$

showing that the conclusions of part (A) hold for $c = c_2$, $h = h_2$ and $c_0 = c_1$.

(c) We emphasize that the assumptions in Theorem 10.3.7 allow for a large class of bounded functions f. ◇

Proof of Theorem 10.3.7. It follows from Section 10.1 that

$$u = g * (f_h \circ u) + r^{-1} * v\,, \tag{10.25}$$

where $g := r^{-1} * s$ and

$$v(n) := \begin{cases} (r * u)(n) - (s * (f_h \circ u))(n)\,, & 0 \le n \le q-1\,, \\ 0\,, & n \ge q\,. \end{cases}$$

Clearly, (10.25) is of the form (10.8): in order to apply Theorem 10.2.1 (with $g = r^{-1} * s$, $\varphi = f_h$, $w = r^{-1} * v$ and $a = 2c$), we need to verify the relevant assumptions. It follows from the hypotheses on ρ and $\mathbb{S}$, via part (a) of Lemma 10.3.5 that (10.2) is strictly zero-stable. The residue of $\hat{g}$ at $z = 1$ is given by $g_0 := \sigma(1)/\rho'(1)$ and, by part (b) of Lemma 10.3.5, $g_0 > 0$. Invoking part (b) of Corollary 10.3.3, we obtain that $g_1 := g - g_0\vartheta \in l^1(\mathbb{Z}_+)$. Consequently, g is of the form required for an application of Theorem 10.2.1. Furthermore, by Lemma 10.3.5

$$1/(2c) + \operatorname{Re}\hat{g}(e^{i\theta}) = 1/(2c) + \operatorname{Re}\left(\sigma(e^{i\theta})/\rho(e^{i\theta})\right) \ge \varepsilon\,, \quad \theta \in (0, 2\pi)\,, \tag{10.26}$$

with $\varepsilon > 0$ under the assumptions of (A) and with $\varepsilon = 0$ under the assumptions of (B). Also note that under the assumptions of (C), (10.26) remains true with $\varepsilon = 0$ and $c = \infty$. Consequently, (10.10) in Theorem 10.2.1 holds under the assumptions of part (A), part (B) and part (C). Next we observe that (10.22) is equivalent to

$$\operatorname{Re}\langle f_h(n,\xi), \xi\rangle \le -1/(2c)\|f_h(n,\xi)\|^2\,, \quad (n,\xi) \in \mathbb{Z}_+ \times U\,. \tag{10.27}$$

Note that for $c = \infty$, inequality (10.27) is equivalent to (10.23). Therefore, $\varphi = f_h$ satisfies the sector condition (10.9) in Theorem 10.2.1. To show that $w = r^{-1} * v$ satisfies the required assumption, we note that by strict zero-stability, combined with statement (b) of Corollary 10.3.3, there exists $r_1 \in l^1(\mathbb{Z}_+)$ such that

$$r^{-1} = \gamma\vartheta + r_1\,, \quad \text{where } \gamma := 1/\rho'(1)\,.$$

Hence

$$w = r^{-1} * v = \gamma\vartheta * v + r_1 * v = \gamma J_0 v + r_1 * v\,.$$

Setting

$$w_2 := \gamma \sum_{k=0}^{q-1} v(k)\,, \quad w_1 := w - w_2\vartheta$$

it is clear that $w_1 \in l^1(\mathbb{Z}_+, U) \subset l^2(\mathbb{Z}_+, U)$. Consequently, $w \in m^1(\mathbb{Z}_+, U) \subset$

$m^2(\mathbb{Z}_+, U)$, so that w satisfies the requirement of Theorem 10.2.1. Furthermore,

$$\|w_2\| \leq |\gamma| \|v\|_{l^1} \leq |\gamma| \sqrt{q} \|v\|_{l^2} \tag{10.28}$$

and

$$\|\gamma\vartheta * v - w_2\vartheta\|_{l^1} \leq |\gamma|(q-1)\|v\|_{l^1}\,, \quad \|\gamma\vartheta * v - w_2\vartheta\|_{l^2} \leq |\gamma|(q-1)\|v\|_{l^2}\,.$$

The last two inequalities imply

$$\|w_1\|_{l^1} \leq (\|r_1\|_{l^1} + |\gamma|(q-1))\|v\|_{l^1}, \;\; \|w_1\|_{l^2} \leq (\max_{\theta\in[0,2\pi)} |\widehat{r_1}(e^{i\theta})| + |\gamma|(q-1))\|v\|_{l^2}. \tag{10.29}$$

To estimate $\|w\|_{m^p}$ in terms of $u(0), u(1), \ldots, u(q-1)$, we consider firstly parts (A) and (B), then part (C).

To prove parts (A) and (B), we use (10.22) to obtain

$$\|v\|_{l^p} \leq L_p \left(\sum_{k=0}^{q-1} \|u(k)\|^p \right)^{1/p}, \quad p = 1, 2, \tag{10.30}$$

where $L_1 := \|r\|_{l^1} + 2c\|s\|_{l^1}$ and

$$L_2 := \max_{\theta\in[0,2\pi)} |\rho(e^{i\theta})| + 2c \max_{\theta\in[0,2\pi)} |\sigma(e^{i\theta})|\,.$$

Inequality (10.30) together with (10.28) yields

$$\|w_2\| \leq L_1|\gamma| \sum_{k=0}^{q-1} \|u(k)\|\,, \quad \|w_2\| \leq L_2|\gamma|\sqrt{q} \left(\sum_{k=0}^{q-1} \|u(k)\|^2 \right)^{1/2}. \tag{10.31}$$

Furthermore, by (10.29)–(10.31),

$$\|w\|_{m^p} \leq M_p \left(\sum_{k=0}^{q-1} \|u(k)\|^p \right)^{1/p}, \quad p = 1, 2,$$

for suitable $M_p > 0$. Parts (A) and (B) follow now from Theorem 10.2.1.

To prove part (C), we estimate

$$\|v\|_{l^1} \leq L \sum_{k=0}^{q-1} (\|u(k)\| + \|hf(kh, u(k))\|)\,, \tag{10.32}$$

where $L := \max(\|r\|_{l^1}, \|s\|_{l^1})$. Therefore, by (10.28),

$$\|w_2\| \leq L|\gamma| \sum_{k=0}^{q-1} (\|u(k)\| + \|hf(kh, u(k))\|) . \tag{10.33}$$

Furthermore, by (10.29), (10.32) and (10.33),

$$\|w\|_{m^1} \leq M \sum_{k=0}^{q-1} (\|u(k)\| + \|hf(kh, u(k))\|) ,$$

for some suitable $M > 0$. Part (C) follows now from Theorem 10.2.1. □

Example 10.3.9. (a) Let ρ and σ be given by $\rho(z) = z - 1$ and $\sigma(z) = 1$ (Euler's method). It is clear that $\mathbb{S} = \overline{\mathbb{B}}(-1, 1)$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(\xi) := \begin{cases} -2\xi\,, & \xi \geq -2\,, \\ 4\,, & \xi < -2\,, \end{cases}$$

so that $f(\xi)/\xi \in \overline{\mathbb{B}}(-1, 1)$ for all $\xi \neq 0$. It follows from part (a) of Remark 10.3.8 that the conclusions of statements (A) of Theorem 10.3.7 hold if $h \in (0, 1)$, whilst statements (B) of the same theorem applies if $h = 1$. In particular, the numerical solution $u : \mathbb{Z}_+ \to U$ converges to 0 as $n \to \infty$ for every choice of $h \in (0, 1)$. If $h = 1$, then $u : \mathbb{Z}_+ \to U$ does not converge in general as $n \to \infty$ (for example, if $u(0) = 2$, then $u(n) = (-1)^n 2$ for all $n \in \mathbb{Z}_+$).
(b) Let ρ and σ be given by $\rho(z) = (3/2)z^2 - 2z + 1/2$ and $\sigma(z) = z^2$ (two-step BDF method). Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(\xi_1, \xi_2) := (-2\xi_1^3 + \xi_2, -\xi_1 - \xi_2^3) .$$

Then $\{z \in \mathbb{C} : \operatorname{Re} z < 0\} \subset \mathbb{S}$ and $\langle f(\xi), \xi \rangle < 0$ for all $\xi \in \mathbb{R}^2 \setminus \{0\}$. Hence, the conclusions of statement (C) of Theorem 10.3.7 hold for every $h > 0$. In particular, the numerical solution $u : \mathbb{Z}_+ \to U$ converges to 0 as $n \to \infty$ for every $h > 0$. ◇

Whilst Theorem 10.3.7 has some overlap with results by Nevanlinna [47, 48], there are also considerable differences: Theorem 10.3.7 assumes ρ to be strictly zero-stable, rather than merely zero-stable as in [47, 48]; on the other hand, numerous aspects of Theorem 10.3.7 are more general than in [47, 48], the assumptions in Theorem 10.3.7 are easier to check than those in [47, 48] and some of the conclusions are stronger as compared to [47, 48]. The following remark gives more details.

Remark 10.3.10. (a) The approach adopted in [47] (with $\theta < 1$) considers f which are independent of t and requires that the map $I - ahf$ is bijective and its inverse is globally Lipschitz [1], where a is a constant which appears in a frequency domain condition involving ρ and σ (in particular $1/a \in \operatorname{int}(\mathbb{S})$), making the application of the main result in [47] potentially awkward if $a \neq 0$ (which, for example, is the case if $\mathbb{S}$ is bounded).

(b) The situation considered in part (C) of Theorem 10.3.7 is dealt with in [47] under the additional assumption that the method is strictly stable at infinity, that is, the roots of σ are contained in the open unit disk $\mathbb{B}(0,1)$.

(c) The constant K in Theorem 10.3.7 does not depend on h in contrast to Theorem 3.1 in [47] (see Remark 3.1 in [47]). Moreover, Theorem 3.1 in [47] does not give estimates for $\|\Delta_0 u\|_{l^2}$, $\|f_h \circ u\|_{l^2}$, $\|\mathrm{Re}\,\langle f_h \circ u, u\rangle\|_{l^1}$ and $\|\sum(f_h \circ u)\|_{l^\infty}$.

(d) Let ρ, σ and f be as in part (a) of Example 10.3.9. Let $a \in \mathbb{R}$ be such that $1/a \in \operatorname{int}(\mathbb{S})$, which holds if and only if $a \in (-\infty, -1/2)$. A routine calculation shows that for every $a \in (-\infty, -1/2)$

$$b := -\inf_{|z|\geq 1} \frac{\rho(z)}{\sigma(z) - a\rho(z)} = 0\,.$$

Set $c := -1/(2a)$. Then, using the notation of [47], $D(a,b) = D(a,0) = \mathbb{B}(-c,c)$. For given $a \in (-\infty,-1/2)$, Theorem 3.1 in [47] applies for all $0 < h < -1/(2a)$. Note that if $h = -1/(2a)$, then

$$(I - ahf)(\xi) = \xi + f(\xi)/2 = 0\,, \quad \xi \geq -2\,,$$

so that $(I - ahf)$ is not invertible. Consequently, Theorem 3.1 in [47] does not apply in this case (see part (a) of this remark). Furthermore, if $a = -1/2$, then $1/a = -2 \notin \operatorname{int}(\mathbb{S})$. It follows that Theorem 3.1 in [47] is not applicable (see pp. 60 in [47]). In particular, Theorem 3.1 in [47] does not give a result for the stepsize $h = 1$.

(e) Using Theorem 10.3.7, it was shown in Example 10.3.9 that (for certain values of h) the numerical solution $u : \mathbb{Z}_+ \to U$ converges to 0 as $n \to \infty$. It seems to be difficult to obtain these convergence properties by applying the results in [48] on the behaviour of u at infinity. In part (a) of Example 10.3.9 the method is not A-stable, whilst A-stability is assumed throughout in [48]. Furthermore, in part (b) of Example 10.3.9 the non-linearity f is not a gradient field, and hence Corollary 2 in [48] (which assumes that the non-linearity is a gradient mapping) cannot be used. $\diamond$

We conclude this chapter by presenting a version of Theorem 10.3.7 which yields

[1] An inspection of the proof of Theorem 3.1 in [47] shows that, if it is the aim to obtain bounds on the numerical solution rather than the error, then the global Lipschitz assumption on $(I - ahf)^{-1}$ can be replaced by a global linear boundedness condition, that is, $\sup_{\xi \in U}(\|(I - ahf)^{-1}\xi\|/\|\xi\|) < \infty$.

stability properties of the difference of two solutions of method (10.2).

Corollary 10.3.11. *Assume that method* (10.2) *satisfies the following two conditions:* $\rho(1) = 0$ *and* $\rho(e^{i\omega}) \neq 0$ *for all* $\omega \in (0, 2\pi)$.

(A) *Assume that there exists* $0 < c < \infty$ *such that*

$$\|hf(nh,\xi_1) - hf(nh,\xi_2) + c(\xi_1 - \xi_2)\| \leq c\|\xi_1 - \xi_2\|, \qquad n \in \mathbb{Z}_+,\ \xi_1, \xi_2 \in U. \tag{10.34}$$

Let $u_1 : \mathbb{Z}_+ \to U$ *and* $u_2 : \mathbb{Z}_+ \to U$ *be solutions of* (10.2). *If* $\mathbb{B}(-c_0, c_0) \subset \mathbb{S}$ *for some* $c_0 > c$, *then there exists a constant* $K > 0$ (*depending only on* c_0, c *and* (ρ, σ)) *such that,*

$$\begin{aligned}&\|u_1 - u_2\|_{l^\infty} + \|\Delta_0(u_1 - u_2)\|_{l^2} \\ &\quad + \|f_h \circ u_1 - f_h \circ u_2\|_{l^2} + \|J_0(f_h \circ u_1 - f_h \circ u_2)\|_{l^\infty} \\ &\quad + (\|\mathrm{Re}\,\langle f_h \circ u_1 - f_h \circ u_2, u_1 - u_2\rangle\|_{l^1})^{1/2} \leq K\left(\sum_{k=0}^{q-1}\|u_1(k) - u_2(k)\|^2\right)^{1/2}.\end{aligned}$$

Furthermore, the limit $\lim_{n\to\infty}\|u_1(n) - u_2(n)\|$ *exists and is finite; in particular, if* $\dim U = 1$, *then* $\lim_{n\to\infty}(u_1(n) - u_2(n))$ *exists.*

(B) *Assume that there exists* $0 < c < \infty$ *such that* (10.34) *holds and further that* $\mathbb{B}(-c, c) \subset \mathbb{S}$. *Let* $u_1 : \mathbb{Z}_+ \to U$ *and* $u_2 : \mathbb{Z}_+ \to U$ *be solutions of* (10.2). *Then, there exists a constant* $K > 0$ (*depending only on* c *and* (ρ, σ)) *such that,*

$$\begin{aligned}&\|u_1 - u_2\|_{l^\infty} + (\|\mathrm{Re}\,\langle f_h \circ u_1 - f_h \circ u_2, u_1 - u_2 + \frac{1}{2c}(f_h \circ u_1 - f_h \circ u_2)\rangle\|_{l^1})^{1/2} \\ &\qquad + \|J_0(f_h \circ u_1 - f_h \circ u_2)\|_{l^\infty} \leq K\sum_{k=0}^{q-1}\|u_1(k) - u_2(k)\|.\end{aligned}$$

(C) *Assume that*

$$\mathrm{Re}\,\langle f(nh,\xi_1) - f(nh,\xi_2), \xi_1 - \xi_2\rangle \leq 0, \quad n \in \mathbb{Z}_+,\ \xi_1, \xi_2 \in U.$$

Let $u_1 : \mathbb{Z}_+ \to U$ *and* $u_2 : \mathbb{Z}_+ \to U$ *be solutions of* (10.2). *If* $\{z \in \mathbb{C} : \mathrm{Re}\, z < 0\} \subset \mathbb{S}$, *then there exists a constant* $K > 0$ (*depending only on* (ρ, σ)) *such that,*

$$\begin{aligned}&\|u_1 - u_2\|_{l^\infty} + |J_0(f_h \circ u_1 - f_h \circ u_2)\|_{l^\infty} \\ &\qquad + (\|\mathrm{Re}\,\langle f_h \circ u_1 - f_h \circ u_2, u_1 - u_2\rangle\|_{l^1})^{1/2} \\ &\qquad \leq K\sum_{k=0}^{q-1}(\|u_1(k) - u_2(k)\| + \|hf(kh, u_1(k)) - hf(kh, u_2(k))\|).\end{aligned}$$

Corollary 10.3.11 follows from an application of Corollary 10.2.2 and arguments similar to those used in the proof of Theorem 10.3.7.

Finally, noting that for f in part (a) of Example 10.3.9, $(f(\xi_1)-f(\xi_2))/(\xi_1-\xi_2) \in \overline{\mathbb{B}}(-1,1)$ for all $\xi_1, \xi_2 \in \mathbb{R}$, $\xi_1 \neq \xi_2$, and, for f in part (b) of Example 10.3.9, $\langle f(\xi_1) - f(\xi_2), \xi_1 - \xi_2 \rangle < 0$ for all $\xi_1, \xi_2 \in \mathbb{R}^2$, $\xi_1 \neq \xi_2$, it follows that a suitably modified version of Remark 10.3.10 holds for Corollary 10.3.11.

10.4 Notes and references

The results in Sections 10.2 and 10.3 together with Lemma 10.1.5 form the basis of [4]. As previously mentioned, Theorem 10.3.7 has some overlap with results by Nevanlinna [47, 48], however, there are considerable differences. One class of new results in this chapter are bounds on $\|u_1(n) - u_2(n)\|$ purely in terms of the initial data; i.e. bounds independent of hf. Additionally, we prove new results on the behaviour of $(u_1(n) - u_2(n))$ as $n \to \infty$ and bounds are obtained for some classes of methods not considered in [47, 48]. For precise details on the differences between Theorem 10.3.7 as compared with the results of [47, 48], see Remark 10.3.10.

Chapter 11

Future work

In this chapter we briefly outline some future research topics related to this thesis.

Consider the discrete-time equation

$$u = r - J_0(G(\varphi \circ u)), \tag{11.1}$$

where $r : \mathbb{Z}_+ \to U$ is a given forcing function, $G \in \mathscr{B}(l^2(\mathbb{Z}_+, U))$ is shift-invariant, $\varphi : \mathbb{Z}_+ \times U \to U$ is a time-dependent non-linearity and $\varphi \circ u$ denotes the function $n \mapsto \varphi(n, u(n))$. Proposition 3.5.3 gives a condition under which (11.1) has at least one solution (a unique solution, respectively), namely, for every $n \in \mathbb{Z}_+$, the map $f_n : U \to U$ defined by

$$f_n(\xi) = \xi + \mathbf{G}(\infty)\varphi(n, \xi), \qquad \forall\ (n, \xi) \in \mathbb{Z}_+ \times U$$

is surjective (bijective, respectively) where $\mathbf{G}(\infty) := \lim_{|z| \to \infty} \mathbf{G}(z)$. Whilst this is a satisfactory condition for the thesis, one possible future research topic is to determine (if possible) sufficient conditions on G and φ which guarantee that the map f_n is surjective (bijective) for every $n \in \mathbb{Z}_+$.

In [9] and [10] continuous-time absolute stability results of Popov-type were obtained in a general Hilbert space setting. In Chapter 4, the corresponding discrete-time absolute stability results of Popov-type are only stated in the special case $U = \mathbb{R}$, this is due to limitations within the proofs of these results, in particular, obtaining a positive lower bound for the term

$$q \sum_{j=0}^{n} (\varphi \circ u)(j)(\Delta_0 u)(j).$$

Consequently, another future research topic is to attempt to extend the Popov-type absolute stability results in Chapter 4 to a possibly infinite-dimensional

Hilbert space setting in line with the continuous-time results in [9] and [10]. Firstly, it is unclear if the Popov-type results in Chapter 4 can be extended to such generality and secondly, to extend these results may require a different approach than the arguments used in Chapter 4.

Another possible extension of the Popov-type absolute stability results in Chapter 4 is to attempt to derive Popov-type results for non-linear operators. A Popov-type result in [15] (see, Theorem on p.192 of [15]), suggests it should be possible to obtain Popov results, with conclusions similar to those in Chapter 4, for causal input-output operators $G : l^2(\mathbb{Z}_+, \mathbb{R}) \to l^2(\mathbb{Z}_+, \mathbb{R})$ that need no longer be linear or shift-invariant. In particular, such input-output operators do not have transfer functions. Consequently, in order to obtain Popov-type results for such non-linear operators we would need to replace the frequency-domain positive real conditions in Chapter 4 with suitable passivity conditions in the time-domain. It is also possible that additional assumptions may need to be imposed on the input-output operator G.

In Theorem 8.1.8 we impose the following assumptions on the time-varying gain $\kappa : \mathbb{Z}_+ \to \mathbb{R}$, namely that κ is bounded and non-negative with

$$\limsup_{n \to \infty} \kappa(n) < 1/|\lambda f_{0,J}(G)| \tag{11.2}$$

where

$$f_{0,J}(G) := \operatorname{ess\,inf}_{\theta \in (0,2\pi)} \mathrm{Re} \left[\frac{\mathbf{G}(e^{i\theta})}{e^{i\theta} - 1} \right]$$

and $\lambda > 0$ denotes a constant related to a static input non-linearity. Here $G := \mathcal{S} G_c \mathcal{H}$ where $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, \mathbb{R}))$ is shift-invariant with transfer function $\mathbf{G}_c$ satisfying assumption (A_c) and $\mathbf{G}_c(0) > 0$. One future research task would be to find an easily computable estimate of the quantity $f_{0,J}(G)$. Moreover, since G is a discrete-time input-output operator, condition (11.2) is not entirely satisfactory in the context of a continuous-time result such as Theorem 8.1.8. An interesting research problem would be to obtain an upper bound C for $|f_{0,J}(G)|$ in terms of G_c and/or $\mathbf{G}_c$. Similar problems, of obtaining easily computable estimates and upper bounds in terms of continuous-time data, can also be considered for the quantities $f_J(G)$, $f_{J_0}(G)$ and $f_{0,J_0}(G)$ (see Chapter 8). Such bounds for the maximal regulating gain have previously been studied for continuous-time systems in [41] and, regular sampled-data systems in [45] (see, Chapter 8, Proposition 8.1.3 of [45]).

Consider the continuous-time low-gain integral control problem shown in Figure 11.1, where $G_c \in \mathscr{B}(L^2(\mathbb{R}_+, \mathbb{R}))$ is shift-invariant with transfer function denoted by $\mathbf{G}_c$, $\varphi : \mathbb{R} \to \mathbb{R}$ is a static input non-linearity, $\rho \in \mathbb{R}$ denotes a constant reference value, g_c models the effect of non-zero initial conditions of the system with input-output operator G_c and $\kappa_c : \mathbb{R}_+ \to \mathbb{R}$ is a time-varying gain. In

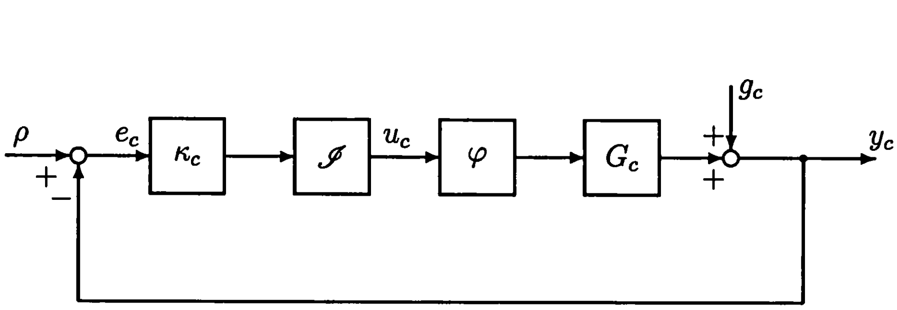

Figure 11.1: Continuous-time low-gain integral control problem

Chapter 8 we introduced the set of feasible reference values

$$\mathscr{R}(G_c, \varphi) := \{\mathbf{G}_c(0)v \mid v \in \overline{\mathrm{im}\varphi}\}.$$

It has been shown in [16] (see, Proposition 3.4 in [16] with $\psi \equiv \mathrm{id}$) that $\rho \in \mathscr{R}(G_c, \varphi)$ is close to being a necessary condition for asymptotic tracking of the error e_c insofar as, if asymptotic tracking of ρ is achievable, whilst maintaining boundedness of $\varphi \circ u_c$ together with ultimate continuity and ultimate boundedness of $g_c + G_c(\varphi \circ u_c)$, then $\rho \in \mathscr{R}(G_c, \varphi)$. We would like to obtain a result similar to that in [16] for the sampled-data feedback system shown in Figure 8.3. However, in the sampled-data context, the continuous-time signal u_c is given by $u_c = \mathcal{H}u$ for some $u \in F(\mathbb{Z}_+, \mathbb{R})$. Consequently, u_c is piecewise continuous, which in turn means we can not obtain ultimate continuity of $g_c + G_c(\varphi \circ u_c)$. Hence the result in [16] can not be applied to the sampled-data systems considered in Chapter 8. An interesting research problem would be to determine (if possible) conditions, similar to those in [16], under which $\rho \in \mathscr{R}(G_c, \varphi)$ is close to being a necessary condition for asymptotic tracking in the sampled-data context.

The continuous-time integral control results in [9] have been extended (see [39]) to include a large class of hysteretic input non-linearities containing many hysteresis operators of relevance in control engineering, such as backlash (play), elastic-plastic (stop) and Preisach operators. Combining results from [34] and [35], the results in Chapters 4-6 and 8-9 could be extended to incorporate hysteretic input non-linearities.

There are several further research topics relating to linear multistep methods. Firstly, it has been seen in Chapter 10 that discrete-time absolute stability theory can be applied to linear multistep methods to obtain stability results. This link with linear multistep methods provides the possibility of using ideas from the stability theory of linear multistep methods in a control theoretic context. In particular reference [49] contains some results which might be useful in absolute stability theory.

Secondly, a result in [3] (see §3.5, Theorem 5.1 of [3]) gives a continuous-time stability result for the following system,

$$u(t) = r(t) + \int_0^t g(t-s)\varphi(u(s))\,\mathrm{d}s, \qquad t \in \mathbb{R}_+,$$

where the non-linearity φ is sector bounded with lower bound possibly zero. It is assumed further that the kernel g takes a certain form namely,

$$g(t) = g_0(t) + \alpha\cos\omega t + \beta\sin\omega t,$$

with $g_0 \in L^1(\mathbb{R}_+, \mathbb{R})$. If we take the Laplace transform of g we see that it has a conjugate pair of poles on the imaginary axis. Due to the presence of a discrete-time integrator in the linear system, the positive real conditions imposed in Chapter 4 have a pole at $z = 1$. It should be possible to prove a corresponding discrete-time stability result analogous to the result in [3], where the linear system has several poles on the unit circle. Such stability results could have applications to low-gain control for sinusoidal reference signals. Results of this type could also be useful for applications to linear multistep stability, in particular, a stability analysis of the so-called 'leapfrog' method (see, for example, [29]).

Chapter 12

Appendices

Appendix 1: Proof of Lemma 4.1.1

Proof of Lemma 4.1.1. We proceed by induction on m. For the case $m = 1$ we have,

$$\begin{aligned}
&\operatorname{Re}\sum_{n=1}^{1}\left\langle v(n),\sum_{k=0}^{0}v(k)\right\rangle \\
&= \operatorname{Re}\langle v(1), v(0)\rangle \\
&= \frac{1}{2}\Big(\langle v(0), v(1)\rangle + \langle v(1), v(0)\rangle\Big) \\
&= \frac{1}{2}\langle v(0)+v(1), v(0)+v(1)\rangle - \frac{1}{2}\Big(\|v(0)\|^2 + \|v(1)\|^2\Big) \\
&= \frac{1}{2}\Big\|v(0)+v(1)\Big\|^2 - \frac{1}{2}\Big(\|v(0)\|^2 + \|v(1)\|^2\Big) \\
&= \frac{1}{2}\Big\|\sum_{k=0}^{1}v(k)\Big\|^2 - \frac{1}{2}\sum_{k=0}^{1}\|v(k)\|^2.
\end{aligned}$$

Assuming that the result is true for m, we obtain,

$$\begin{aligned}
&\operatorname{Re}\sum_{n=1}^{m+1}\left\langle v(n),\sum_{k=0}^{n-1}v(k)\right\rangle \\
&= \operatorname{Re}\left\langle v(m+1),\sum_{k=0}^{m}v(k)\right\rangle + \frac{1}{2}\Big\|\sum_{k=0}^{m}v(k)\Big\|^2 - \frac{1}{2}\sum_{k=0}^{m}\|v(k)\|^2
\end{aligned}$$

$$= \frac{1}{2}\|v(m+1)\|^2 - \frac{1}{2}\|v(m+1)\|^2 + \mathrm{Re}\left\langle v(m+1), \sum_{k=0}^{m} v(k)\right\rangle$$
$$+\frac{1}{2}\left\|\sum_{k=0}^{m} v(k)\right\|^2 - \frac{1}{2}\sum_{k=0}^{m}\|v(k)\|^2$$
$$= \frac{1}{2}\left\|\sum_{k=0}^{m+1} v(k)\right\|^2 - \frac{1}{2}\sum_{k=0}^{m+1}\|v(k)\|^2,$$

advancing the inductive argument from m to $m+1$. Hence the claim is true for all $m \in \mathbb{N}$ by induction. □

Appendix 2: Upper and lower bounds on the quantities $f_J(G)$, $f_{J_0}(G)$, $f_{0,J}(G)$ and $f_{0,J_0}(G)$

Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ be shift-invariant and $\mathbf{G} \in H^\infty(\mathbb{E}_1)$ denote the transfer function of G. Recall that,

$$f_J(G) := \sup_{q \geq 0}\left\{\operatorname{ess\,inf}_{\theta\in(0,2\pi)}\mathrm{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right]\right\}$$

and

$$f_{0,J}(G) := \operatorname{ess\,inf}_{\theta\in(0,2\pi)}\mathrm{Re}\left[\frac{\mathbf{G}(e^{i\theta})}{e^{i\theta}-1}\right].$$

We require the following lemma.

Lemma 12.1.1. *Let $f \in H^\infty(\mathbb{E}_1)$. Then,*

$$\operatorname{ess\,inf}_{\theta\in(0,2\pi)}\mathrm{Re}\, f(e^{i\theta}) \leq \mathrm{Re}\, f(\infty),$$

where $f(\infty) := \lim_{|z|\to\infty, z\in\mathbb{E}_1} f(z)$.

Remark 12.1.2. Suppose that $G \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ is shift-invariant with transfer function $\mathbf{G} \in H^\infty(\mathbb{E}_1)$. Let $z \in \mathbb{E}_1 \cap \mathbb{R}$. Then $\mathbf{G}(z) \in \mathbb{R}$. To see this we first note that if $u \in l^2(\mathbb{Z}_+, \mathbb{R})$, then $y = Gu \in l^2(\mathbb{Z}_+, \mathbb{R})$. Consequently,

$$\widehat{y}(\overline{z}) = \sum_{j=0}^{\infty} y(j)\overline{z}^{-j} = \mathbf{G}(\overline{z})\sum_{j=0}^{\infty} u(j)\overline{z}^{-j}, \quad z \in \mathbb{E}_1.$$

Furthermore,

$$\overline{\widehat{y}(z)} = \sum_{j=0}^{\infty} y(j)\overline{z}^{-j} = \overline{\mathbf{G}(z)} \sum_{j=0}^{\infty} u(j)\overline{z}^{-j}, \quad z \in \mathbb{E}_1,$$

showing that $\mathbf{G}(\overline{z}) = \overline{\mathbf{G}(z)}$. Hence, if $z \in \mathbb{E}_1 \cap \mathbb{R}$, then we have $\mathbf{G}(z) = \overline{\mathbf{G}(z)}$ showing that $\mathbf{G}(z) \in \mathbb{R}$. In particular, since $\mathbf{G}(z) \in \mathbb{R}$ for all $z \in \mathbb{E}_1 \cap \mathbb{R}$, if $\mathbf{G}(1) := \lim_{z \to 1, z \in \mathbb{E}_1} \mathbf{G}(z)$ and $\mathbf{G}(\infty) := \lim_{|z| \to \infty, z \in \mathbb{E}_1} \mathbf{G}(z)$ exist, then $\mathbf{G}(1), \mathbf{G}(\infty) \in \mathbb{R}$. ◇

Proof of Lemma 12.1.1. Setting $\widetilde{f} := \operatorname{ess\,inf}_{\theta \in (0,2\pi)} \operatorname{Re} f(e^{i\theta})$, we have

$$e^{-\operatorname{Re} f(\infty)} \leq \sup_{z \in \mathbf{E}_1} |e^{-f(z)}| = \operatorname{ess\,sup}_{\theta \in (0,2\pi)} |e^{-f(e^{i\theta})}| = e^{-\widetilde{f}},$$

where the first equality follows from the fact that $e^{-f} \in H^\infty(\mathbb{E}_1)$ and Theorem 3.3.2. Consequently, $\widetilde{f} \leq \operatorname{Re} f(\infty)$ as required. □

In the following we assume that the transfer function $\mathbf{G}$ of G satisfies assumption (A). We introduce the following auxiliary transfer function

$$\mathbf{H}(z) := \frac{\mathbf{G}(z) - \mathbf{G}(1)}{z - 1}, \quad z \in \mathbb{E}_1, \tag{A.1}$$

where, by Remark 12.1.2, $\mathbf{G}(1) \in \mathbb{R}$. The following result gives upper and lower bounds on $f_J(G)$ and $f_{0,J}(G)$.

Proposition 12.1.3. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ be shift-invariant with transfer function $\mathbf{G}$ of G satisfying assumption* (A). *Then,*

(i) $-\infty < f_J(G) \leq -\mathbf{G}(1)/2$,

(ii) $-\infty < f_{0,J}(G) \leq -\mathbf{G}(1)/2$.

Proof. We first note that it follows trivially from the definitions of $f_J(G)$ and $f_{0,J}(G)$ that

$$f_J(G) \geq f_{0,J}(G). \tag{A.2}$$

We claim that $f_{0,J}(G) > -\infty$. From assumption (A) and the fact that $\mathbf{G} \in H^\infty(\mathbb{E}_1)$, we have that $\mathbf{H} \in H^\infty(\mathbb{E}_1)$. Hence it follows that,

$$\begin{aligned} f_{0,J}(G) &= \operatorname{ess\,inf}_{\theta \in (0,2\pi)} \operatorname{Re} \left[\frac{\mathbf{G}(e^{i\theta}) - \mathbf{G}(1)}{e^{i\theta} - 1} + \frac{\mathbf{G}(1)}{e^{i\theta} - 1} \right] \\ &= \operatorname{ess\,inf}_{\theta \in (0,2\pi)} \operatorname{Re} \mathbf{H}(e^{i\theta}) - \frac{\mathbf{G}(1)}{2} \\ &> -\infty, \end{aligned}$$

where the final inequality follows from the fact that $\mathbf{H} \in H^\infty(\mathbb{E}_1)$. Hence $f_{0,J}(G) > -\infty$ and it follows from (A.2) that $f_J(G) > -\infty$. By (A.2), in order to show that (i) and (ii) hold, it remains to show that $f_J(G) \leq -\mathbf{G}(1)/2$. To this end, we first set

$$\mathbf{G}_q(z) := q\frac{\mathbf{G}(z)}{z} + \frac{\mathbf{G}(z) - \mathbf{G}(1)}{z-1}, \qquad z \in \mathbb{E}_1.$$

Then $\mathbf{G}_q \in H^\infty(\mathbb{E}_1)$ and $\mathbf{G}_q(\infty) = 0$. By Lemma 12.1.1,

$$\operatorname{ess\,inf}_{\theta\in(0,2\pi)} \operatorname{Re} \mathbf{G}_q(e^{i\theta}) \leq 0.$$

Now,

$$\operatorname{Re} \mathbf{G}_q(e^{i\theta}) = \operatorname{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right] + \frac{\mathbf{G}(1)}{2}.$$

Hence

$$\operatorname{ess\,inf}_{\theta\in(0,2\pi)} \operatorname{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right] \leq -\frac{\mathbf{G}(1)}{2}, \qquad \forall\, q \geq 0,$$

implying that

$$f_J(G) = \sup_{q\geq 0}\left\{\operatorname{ess\,inf}_{\theta\in(0,2\pi)} \operatorname{Re}\left[\left(\frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right]\right\} \leq -\frac{\mathbf{G}(1)}{2},$$

as required. □

Recall that,

$$f_{J_0}(G) := \sup_{q\geq 0}\left\{\operatorname{ess\,inf}_{\theta\in(0,2\pi)} \operatorname{Re}\left[\left(q + \frac{e^{i\theta}}{e^{i\theta}-1}\right)\mathbf{G}(e^{i\theta})\right]\right\}$$

and

$$f_{0,J_0}(G) := \operatorname{ess\,inf}_{\theta\in(0,2\pi)} \operatorname{Re}\left[\frac{\mathbf{G}(e^{i\theta})e^{i\theta}}{e^{i\theta}-1}\right].$$

Proposition 12.1.4. *Let $G \in \mathscr{B}(l^2(\mathbb{Z}_+, \mathbb{R}))$ be shift-invariant with transfer function $\mathbf{G}$ of G satisfying assumption* (A). *Then,*

(i) $-\infty < f_{J_0}(G) \leq \infty$,

(ii) $-\infty < f_{0,J_0}(G) \leq \mathbf{G}(\infty) - \mathbf{G}(1)/2$.

Proof. We first note that it follows trivially from the definitions of $f_{J_0}(G)$ and $f_{0,J_0}(G)$ that

$$f_{J_0}(G) \geq f_{0,J_0}(G). \tag{A.3}$$

We claim that $f_{0,J_0}(G) > -\infty$. From assumption (A) and the fact that $\mathbf{G} \in H^\infty(\mathbb{E}_1)$, we have that $z \mapsto z\mathbf{H}(z) \in H^\infty(\mathbb{E}_1)$, where $z \in \mathbb{E}_1$ and $\mathbf{H}$ is given by (A.1). Hence it follows that,

$$\begin{aligned} f_{0,J_0}(G) &= \operatorname{ess\,inf}_{\theta\in(0,2\pi)} \mathrm{Re}\left[\frac{(\mathbf{G}(e^{i\theta}) - \mathbf{G}(1))e^{i\theta}}{e^{i\theta}-1} + \frac{\mathbf{G}(1)e^{i\theta}}{e^{i\theta}-1}\right] \\ &= \operatorname{ess\,inf}_{\theta\in(0,2\pi)} \mathrm{Re}\,(e^{i\theta}\mathbf{H}(e^{i\theta})) + \frac{\mathbf{G}(1)}{2} \\ &> -\infty, \end{aligned}$$

where the final inequality follows from the fact that $z \mapsto z\mathbf{H}(z) \in H^\infty(\mathbb{E}_1)$. Hence $f_{0,J_0}(G) > -\infty$ and it follows from (A.3) that $f_{J_0}(G) > -\infty$. Note that trivially $f_{J_0}(G) \leq \infty$, which, together with the fact that $f_{J_0}(G) > -\infty$, shows that (i) holds.

Define now

$$\widetilde{\mathbf{H}}(z) := z\mathbf{H}(z), \qquad z \in \mathbb{E}_1.$$

Then $\widetilde{\mathbf{H}} \in H^\infty(\mathbb{E}_1)$ and

$$\widetilde{\mathbf{H}}(\infty) = \mathbf{G}(\infty) - \mathbf{G}(1).$$

By Lemma 12.1.1,

$$\operatorname{ess\,inf}_{\theta\in(0,2\pi)} \mathrm{Re}\,\widetilde{\mathbf{H}}(e^{i\theta}) \leq \mathbf{G}(\infty) - \mathbf{G}(1).$$

Now,

$$\mathrm{Re}\,\widetilde{\mathbf{H}}(e^{i\theta}) = \mathrm{Re}\left[\frac{\mathbf{G}(e^{i\theta})e^{i\theta}}{e^{i\theta}-1}\right] - \frac{\mathbf{G}(1)}{2}.$$

Hence

$$\operatorname{ess\,inf}_{\theta\in(0,2\pi)} \mathrm{Re}\left[\frac{\mathbf{G}(e^{i\theta})e^{i\theta}}{e^{i\theta}-1}\right] \leq \mathbf{G}(\infty) - \frac{\mathbf{G}(1)}{2},$$

showing that $f_{0,J_0}(G) \leq \mathbf{G}(\infty) - \mathbf{G}(1)/2$, which, together with the fact that $f_{0,J_0}(G) > -\infty$, shows that (ii) holds. $\square$

Appendix 3: Details in Remark 5.1.3 (iii) and (iv)

Proposition 12.1.5. *Let $f \in F(\mathbb{Z}_+, \mathbb{R})$. Then $Jf \in m^2(\mathbb{Z}_+, \mathbb{R})$ if and only if $(Jf)(n)$ converges to a finite limit as $n \to \infty$ and $n \mapsto \sum_{k=n}^{\infty} f(k)$ is in $l^2(\mathbb{Z}_+, \mathbb{R})$.*

Proof. Assume $(Jf)(n) \to \sigma \in \mathbb{R}$ as $n \to \infty$. Set $h(n) := \sum_{k=n}^{\infty} f(k)$ for all

$n \in \mathbb{Z}_+$ and assume that $h \in l^2(\mathbb{Z}_+, \mathbb{R})$. Then

$$(Jf)(n) + h(n) = \sigma, \qquad \forall\, n \in \mathbb{Z}_+,$$

showing that $Jf \in m^2(\mathbb{Z}_+, \mathbb{R})$. Conversely, if $Jf \in m^2(\mathbb{Z}_+, \mathbb{R})$, then

$$(Jf)(n) = h(n) + \sigma, \qquad \forall\, n \in \mathbb{Z}_+, \tag{A.4}$$

where $\sigma \in \mathbb{R}$ and $h \in l^2(\mathbb{Z}_+, \mathbb{R})$. Therefore $h(n) \to 0$ as $n \to \infty$ and so

$$(Jf)(n) \to \sigma = \sum_{k=0}^{\infty} f(k).$$

Hence, by (A.4)

$$\sum_{k=n}^{\infty} f(k) = -h(n)$$

completing the proof. □

Proposition 12.1.6. *Let $f \in F(\mathbb{Z}_+, \mathbb{R})$ be such that $j \mapsto f(j)j^\alpha$ is in $l^2(\mathbb{Z}_+, \mathbb{R})$ for some $\alpha > 1$, then $n \mapsto \sum_{j=n}^{\infty} |f(j)| \in l^2(\mathbb{Z}_+, \mathbb{R})$.*

Proof. Let $1/2 < \beta \leq \alpha/2$. Then

$$\sum_{j=n}^{\infty} |f(j)| = n^{-\beta} n^{\beta} \sum_{j=n}^{\infty} |f(j)| \leq n^{-\beta} \sum_{j=n}^{\infty} |f(j)| j^{\beta} = n^{-\beta} \sum_{j=n}^{\infty} |f(j)| j^{\alpha} j^{\beta-\alpha}, \quad n \geq 1.$$

Since $\beta - \alpha \leq -\alpha/2 < -1/2$ it follows that $j \mapsto j^{\beta-\alpha} \in l^2(\mathbb{Z}_+, \mathbb{R})$. By assumption $j \mapsto f(j)j^\alpha \in l^2(\mathbb{Z}_+, \mathbb{R})$. Consequently, by Hölders inequality,

$$\begin{aligned}
\sum_{j=n}^{\infty} |f(j)| &\leq n^{-\beta} \sum_{j=n}^{\infty} |f(j)| j^{\alpha} j^{\beta-\alpha} \\
&\leq n^{-\beta} \left(\sum_{j=1}^{\infty} |f(j)|^2 j^{2\alpha} \right)^{1/2} \left(\sum_{j=1}^{\infty} j^{2(\beta-\alpha)} \right)^{1/2} \\
&\leq \gamma n^{-\beta}, \quad n \geq 1,
\end{aligned}$$

where $\gamma := (\sum_{j=1}^{\infty} |f(j)|^2 j^{2\alpha})^{1/2} (\sum_{j=1}^{\infty} j^{2(\beta-\alpha)})^{1/2} < \infty$. Note that since $j \mapsto f(j)j^\alpha \in l^2(\mathbb{Z}_+, \mathbb{R})$, it follows from Hölders inequality that,

$$\sum_{j=n}^{\infty} |f(j)| = \sum_{j=n}^{\infty} |f(j)| j^{\alpha} j^{-\alpha} \leq \left(\sum_{j=0}^{\infty} |f(j)|^2 j^{2\alpha} \right)^{1/2} \left(\sum_{j=n}^{\infty} j^{-2\alpha} \right)^{1/2}, \quad \forall\, n \geq 1.$$

Hence, $f \in l^1(\mathbb{Z}_+, \mathbb{R})$ and we have

$$\sum_{n=0}^{\infty}\left(\sum_{j=n}^{\infty}|f(j)|\right)^2 \leq \|f\|_{l^1}^2 + \gamma\sum_{n=1}^{\infty} n^{-2\beta}.$$

Since $2\beta > 1$, it follows that $\sum_{n=0}^{\infty} n^{-2\beta} < \infty$, showing that $n \mapsto \sum_{j=n}^{\infty}|f(j)| \in l^2(\mathbb{Z}_+, \mathbb{R})$. □

Appendix 4: Proof of Proposition 5.2.1

Proof of Proposition 5.2.1. By assumption, we have the following state-space realisation of $\mathbf{G}$,

$$\mathbf{G}(z) = C(zI - A)^{-1}B + D, \qquad z \in \mathbb{E}_\alpha$$

where $0 < \alpha < 1$, $A \in \mathscr{B}(X)$ is power stable, $B \in \mathscr{B}(\mathbb{R}, X)$, $C \in \mathscr{B}(X, \mathbb{R})$, $D \in \mathbb{R}$ and X denotes a Hilbert space. Defining $x \in F(\mathbb{Z}_+, X)$ by,

$$x(n+1) = Ax(n) + B(\varphi \circ u)(n), \qquad x(0) = 0, \tag{A.5}$$

it follows that

$$w = g + Cx + D(\varphi \circ u).$$

Since $\varphi \circ u \in l^\infty(\mathbb{Z}_+, \mathbb{R})$, it follows from Lemma 6.1.7 that x is bounded. Furthermore,

$$w = g + C(x - (I - A)^{-1}B(\varphi \circ u)) + \mathbf{G}(1)(\varphi \circ u).$$

Note that since, by assumption, $g(n) \to 0$ as $n \to \infty$ we have $g \in l^\infty(\mathbb{Z}_+, \mathbb{R})$. Combining the previous it is clear that $w \in l^\infty(\mathbb{Z}_+, \mathbb{R})$. Since $w \in l^\infty(\mathbb{Z}_+, \mathbb{R})$, it has a non-empty, compact ω-limit set Ω. By continuity of ψ and (5.18), it follows that

$$\psi(\omega) = \rho, \qquad \forall\, \omega \in \Omega. \tag{A.6}$$

Since ψ is monotone and continuous, it follows that $\psi^{-1}(\rho)$ is a closed interval. It suffices to show that $\psi^{-1}(\rho) \cap \mathbf{G}(1)\overline{\mathrm{im}\varphi} \neq \emptyset$. Indeed if there exists $\xi \in \psi^{-1}(\rho)$ and $v \in \overline{\mathrm{im}\varphi}$ such that $\xi = \mathbf{G}(1)v$, then $\psi(\mathbf{G}(1)v) = \psi(\xi) = \rho$, showing that $\rho \in \mathscr{R}(G, \varphi, \psi)$.

Seeking a contradiction, suppose that $\psi^{-1}(\rho) \cap \mathbf{G}(1)\overline{\mathrm{im}\varphi} = \emptyset$. Then $\psi^{-1}(\rho)$ and $\overline{\mathrm{im}\varphi}$ are closed disjoint intervals and so there exist $\varepsilon > 0$ and $\beta = \pm 1$ such that

$$\inf\{\beta\xi \mid \xi \in \psi^{-1}(\rho)\} - \sup\{\beta\mathbf{G}(1)v \mid v \in \overline{\mathrm{im}\varphi}\} = 2\varepsilon.$$

Moreover, since $w(n)$ approaches its compact ω-limit set Ω as $n \to \infty$, and, by

(A.6), $\Omega \subseteq \psi^{-1}(\rho)$, it follows that,

$$\lim_{n\to\infty} \operatorname{dist}(\beta w(n), \beta\psi^{-1}(\rho)) = 0,$$

and so there exists $N > 0$ such that $\beta w(n) > \inf\{\beta\xi \mid \xi \in \psi^{-1}(\rho)\} - \varepsilon$ for all $n > N$. Consequently,

$$\begin{aligned}\beta(w(n) - \mathbf{G}(1)(\varphi \circ u)(n)) &\geq \beta w(n) - \sup\{\beta\mathbf{G}(1)v \mid v \in \overline{\operatorname{im}\varphi}\} \\ &> \inf\{\beta\xi \mid \xi \in \psi^{-1}(\rho)\} - \varepsilon - \sup\{\beta\mathbf{G}(1)v \mid v \in \overline{\operatorname{im}\varphi}\} \\ &= \varepsilon, \qquad \forall\, n > N. \end{aligned} \tag{A.7}$$

By (A.5) it follows that,

$$(I - A)^{-1}x(n+1) = (I - A)^{-1}Ax(n) + (I - A)^{-1}B(\varphi \circ u)(n). \tag{A.8}$$

Noting that $(I - A)^{-1}A = (I - A)^{-1} - I$ we obtain from (A.8),

$$(I - A)^{-1}(\Delta x)(n) = -(x(n) - (I - A)^{-1}B(\varphi \circ u)(n)).$$

Consequently,

$$C(I - A)^{-1}(\Delta x)(n) = g(n) + \mathbf{G}(1)(\varphi \circ u)(n) - w(n).$$

For $n, N \in \mathbb{Z}_+$ with $n > N$, summing the above from N to $n-1$ gives

$$C(I - A)^{-1}(x(n) - x(N)) = \sum_{k=N}^{n-1}(g(k) + \mathbf{G}(1)(\varphi \circ u)(k) - w(k)).$$

Since $g(n) \to 0$ as $n \to \infty$ there exists $M \in \mathbb{N}$ such that $|g(n)| \leq \varepsilon/2$ for all $n \geq M$. Choosing $K \geq \max(N, M)$ and using (A.7), we have that,

$$\begin{aligned}-\beta C(I - A)^{-1}(x(n) - x(K)) &= \sum_{k=K}^{n-1} \beta(w(k) - \mathbf{G}(1)(\varphi \circ u)(k)) - \beta g(k) \\ &\geq \left(\varepsilon - \frac{\varepsilon}{2}\right)(n - K) \\ &= \frac{(n-K)}{2}\varepsilon, \quad \forall\, n > K.\end{aligned}$$

Therefore $\lim_{n\to\infty} \beta C(A - I)^{-1}x(n) = \infty$, contradicting the boundedness of x. □

Remark 12.1.7. The proof of Proposition 5.2.1 is based on a modification of the argument used in establishing the continuous-time result Proposition 3.4 of [16]. ◇

Appendix 5: Proof of Lemmas 6.1.5, 6.1.6 and 6.1.7

Proof of Lemma 6.1.5. By (6.2) we have,

$$\|x(n)\| \leq \|A^n\|\|x^0\| + \left\| \sum_{j=0}^{n-1} A^{(n-1)-j} B v(j) \right\|, \qquad n \geq 1. \tag{A.9}$$

Defining

$$\widetilde{v}(j) := \begin{cases} v((n-1)-j), & \text{if } 0 \leq j \leq n-1, \\ 0, & \text{if } j \geq n, \end{cases}$$

it follows from (6.4) that,

$$\left\| \sum_{j=0}^{n-1} A^{(n-1)-j} B v(j) \right\| = \left\| \sum_{j=0}^{\infty} A^j B \widetilde{v}(j) \right\| \leq \alpha \|\widetilde{v}\|_{l^2} \leq \alpha \|v\|_{l^2}, \quad n \geq 1, \tag{A.10}$$

for some $\alpha \geq 0$. Noting that A is strongly stable, an application of the uniform boundedness principle to $\{A^n\}_{n \in \mathbb{N}}$ yields the existence of a constant $K_1 \geq 0$ such that $\|A^n\| \leq K_1$ for all $n \in \mathbb{N}$. Combining this fact with (A.10) and (A.9) yields

$$\|x(n)\| \leq K_1 \|x^0\| + \alpha \|v\|_{l^2}, \qquad \forall\, n \in \mathbb{Z}_+.$$

Consequently, with $K := \max\{K_1, \alpha\}$,

$$\|x\|_{l^\infty} \leq K(\|x^0\| + \|v\|_{l^2}).$$

It remains to show that $\lim_{n \to \infty} x(n) = 0$. To this end note that by (6.1a) we have, for $m \in \mathbb{Z}_+$,

$$x(n) = A^{n-m} x(m) + \sum_{j=m}^{n-1} A^{(n-1)-j} B v(j), \qquad \forall\, n \geq m+1. \tag{A.11}$$

Defining

$$\widetilde{v}(j) := \begin{cases} v((n-1)-j), & 0 \leq j \leq n-m-1, \\ 0, & \text{otherwise}, \end{cases}$$

it follows that

$$\sum_{j=m}^{n-1} A^{(n-1)-j} B v(j) = \sum_{j=0}^{n-m-1} A^j B \widetilde{v}(j), \quad \forall\, n \geq m+1.$$

Consequently, invoking (6.4) we obtain,

$$\begin{aligned}\left\|\sum_{j=m}^{n-1} A^{(n-1)-j}Bv(j)\right\| &= \left\|\sum_{j=0}^{n-m-1} A^{j}B\widetilde{v}(j)\right\| \\ &\leq \alpha\left(\sum_{j=m}^{n-1}\|v(j)\|^2\right)^{1/2}, \qquad n \geq m+1, \qquad \text{(A.12)}\end{aligned}$$

for some $\alpha \geq 0$. Let $\varepsilon > 0$. Since $v \in l^2(\mathbb{Z}_+, U)$, there exists $m \geq 0$ such that

$$\sum_{j=m}^{n-1}\|v(j)\|^2 \leq \sum_{j=m}^{\infty}\|v(j)\|^2 \leq \frac{\varepsilon^2}{4\alpha^2}, \quad n \geq m+1. \qquad \text{(A.13)}$$

By the strong stability of A there exists $m_1 \geq 0$ such that

$$\|A^n x(m)\| \leq \varepsilon/2, \qquad \forall\, n \geq m_1. \qquad \text{(A.14)}$$

Hence combining (A.12) - (A.14) with (A.11), we obtain for all $n > m + m_1$

$$\|x(n)\| \leq \|A^{n-m}x(m)\| + \left\|\sum_{j=m}^{n-1} A^{(n-1)-j}Bv(j)\right\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

showing that $\lim_{n\to\infty}\|x(n)\| = 0$. □

Proof of Lemma 6.1.6. Using the fact that $1 \in \operatorname{res}(A)$, it follows from (6.1a) that,

$$(I-A)^{-1}x(n+1) = (I-A)^{-1}Ax(n) + (I-A)^{-1}Bv(n). \qquad \text{(A.15)}$$

Noting that $(I-A)^{-1}A = (I-A)^{-1} - I$ we obtain from (A.15),

$$(I-A)^{-1}(\Delta x)(n) = -(x(n) - (I-A)^{-1}Bv(n)). \qquad \text{(A.16)}$$

Again by (6.1a)

$$(\Delta x)(n+1) = A(\Delta x)(n) + B(\Delta v)(n). \qquad \text{(A.17)}$$

Since by assumption $\Delta v \in l^2(\mathbb{Z}_+, U)$, an application of Lemma 6.1.5 to (A.17) yields $\lim_{n\to\infty}(\Delta x)(n) = 0$. Consequently, taking the limit as $n \to \infty$ in (A.16) we obtain

$$\lim_{n\to\infty}(x(n) - (I-A)^{-1}Bv(n)) = 0.$$

□

Proof of Lemma 6.1.7. Assume that A is power stable and let $v \in l^\infty(\mathbb{Z}_+, U)$. By power stability of A,

$$\|A^n\| \le K_1 \rho^n, \qquad \forall\, n \in \mathbb{Z}_+,$$

for some $K_1 \ge 1$ and $0 < \rho < 1$. Consequently, $\|A^n\| \le K_1$ for all $n \in \mathbb{N}$. Hence from (6.2) we obtain,

$$\begin{aligned} \|x(n)\| &\le \|A^n x^0\| + \left\| \sum_{j=0}^{n-1} A^{(n-1)-j} B v(j) \right\| \\ &\le K_1 \|x^0\| + \left\| \sum_{j=0}^{n-1} A^{(n-1)-j} B v(j) \right\|, \qquad n \ge 1. \end{aligned}$$

Consequently,

$$\sup_{n\ge 1} \|x(n)\| \le K_1 \|x^0\| + \sup_{n\ge 1} \left\| \sum_{j=0}^{n-1} A^{(n-1)-j} B v(j) \right\|. \tag{A.18}$$

Now by power stability of A, there exists an $M \ge 1$ and $\gamma \in (0,1)$ such that

$$\begin{aligned} \sup_{n\ge 1} \left\| \sum_{j=0}^{n-1} A^{(n-1)-j} B v(j) \right\| &\le \sup_{n\ge 1} \|v(n)\| \|B\| \sup_{n\ge 1} \sum_{j=0}^{n-1} \|A^{(n-1)-j}\| \\ &\le \|v\|_{l^\infty} \|B\| \left(\sum_{j=0}^{\infty} \|A^j\| \right) \\ &\le M \|v\|_{l^\infty} \|B\| \left(\sum_{j=0}^{\infty} \gamma^j \right) \\ &= K_2 \|v\|_{l^\infty}, \end{aligned} \tag{A.19}$$

where $K_2 := M\|B\|/(1-\gamma)$. Combining (A.18) and (A.19) yields

$$\sup_{n\ge 1} \|x(n)\| \le K(\|x^0\| + \|v\|_{l^\infty}), \tag{A.20}$$

where $K := \max\{K_1, K_2\} \ge 1$. Consequently, we see from (A.20) that $x \in l^\infty(\mathbb{Z}_+, X)$ and

$$\|x\|_{l^\infty} \le K(\|x^0\| + \|v\|_{l^\infty}).$$

Suppose now that $\lim_{n\to\infty} v(n) = v^\infty$ exists. To see that $\lim_{n\to\infty} x(n) = (I -$

$A)^{-1}Bv^\infty$, we observe the following. Note that by (6.2) we obtain, for all $n \geq 1$,

$$\begin{aligned} x(n) &= A^n x^0 + \sum_{j=0}^{n-1} A^{(n-1)-j} B(v(j) - v^\infty) + \sum_{j=0}^{n-1} A^{(n-1)-j} B v^\infty \\ &= A^n x^0 + \sum_{j=0}^{n-1} A^{(n-1)-j} B(v(j) - v^\infty) + (I-A)^{-1} B v^\infty - A^n (I-A)^{-1} B v^\infty. \end{aligned}$$

Hence it follows that,

$$x(n) - (I-A)^{-1}Bv^\infty = A^n(x^0 - (I-A)^{-1}Bv^\infty) + \sum_{j=0}^{n-1} A^{(n-1)-j} B(v(j) - v^\infty), \quad n \geq 1.$$

Consequently, with $z := x - (I-A)^{-1}Bv^\infty\vartheta$ and $w := v - v^\infty\vartheta$, we have,

$$z(n) = A^n z^0 + \sum_{j=0}^{n-1} A^{(n-1)-j} B w(j), \qquad n \geq 1,$$

where $z(0) = z^0$. Hence, for $m \in \mathbb{Z}_+$,

$$z(n) = A^{n-m} z(m) + \sum_{j=m}^{n-1} A^{(n-1)-j} B w(j), \qquad \forall\, n \geq m+1. \tag{A.21}$$

Now by power stability of A, there exists an $M \geq 1$ and $\gamma \in (0,1)$ such that,

$$\begin{aligned} \left\| \sum_{j=m}^{n-1} A^{(n-1)-j} B w(j) \right\| &\leq \sup_{m \leq j} \|w(j)\| \|B\| \left(\sum_{j=0}^{\infty} \|A^j\| \right) \\ &\leq \sup_{m \leq j} \|w(j)\| \|B\| M \left(\sum_{j=0}^{\infty} \gamma^j \right) \\ &= \sup_{m \leq j} \|w(j)\| \|B\| \frac{M}{1-\gamma} \\ &= \alpha \sup_{m \leq j} \|w(j)\|, \qquad n \geq m+1, \end{aligned} \tag{A.22}$$

where $\alpha := M\|B\|/(1-\gamma)$. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} w(n) = \lim_{n\to\infty}(v(n) - v^\infty) = 0$, there exists $m \geq 0$ such that

$$\|w(j)\| \leq \frac{\varepsilon}{2\alpha}, \qquad \forall\, j \geq m.$$

Hence by (A.22) we see that,

$$\left\| \sum_{j=m}^{n-1} A^{(n-1)-j} Bw(j) \right\| \leq \frac{\varepsilon}{2}, \qquad \forall\, n \geq m+1. \tag{A.23}$$

Since A is power stable, A is strongly stable and there exists $m_1 \geq 0$ such that

$$\|A^n z(m)\| \leq \frac{\varepsilon}{2}, \qquad \forall\, n \geq m_1. \tag{A.24}$$

Hence combining (A.23) and (A.24) with (A.21), we see that for all $n > m + m_1$,

$$\|z(n)\| \leq \|A^{n-m} z(m)\| + \left\| \sum_{j=m}^{n-1} A^{(n-1)-j} Bw(j) \right\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

showing that $\lim_{n\to\infty} \|z(n)\| = 0$, or, equivalently, $\lim_{n\to\infty} x(n) = (I - A)^{-1} Bv^\infty$. $\square$

Appendix 6: Proof of Lemmas 9.1.4 and 9.1.5

Proof of Lemma 9.1.4. Let $x^0 \in X$, $v^\infty \in \mathbb{R}$ be such that $v - v^\infty \vartheta_c \in L^2(\mathbb{R}_+, \mathbb{R})$ and assume that $\mathbf{T}$ is strongly stable. Note that for all $t \in \mathbb{R}_+$,

$$\begin{aligned}
x(t) &= \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-s} Bv(s)\, ds \\
&= \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-s} B(v(s) - v^\infty)\, ds + \int_0^t \mathbf{T}_{t-s} Bv^\infty\, ds \\
&= \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-s} B(v(s) - v^\infty)\, ds + (\mathbf{T}_t - I) A^{-1} Bv^\infty \\
&= \mathbf{T}_t (x^0 + A^{-1} Bv^\infty) + \int_0^t \mathbf{T}_{t-s} B(v(s) - v^\infty)\, ds - A^{-1} Bv^\infty. \qquad \text{(A.25)}
\end{aligned}$$

Set $z := x + A^{-1} Bv^\infty \vartheta_c$ and $w := v - v^\infty \vartheta_c$. Then by (A.25) we have

$$\begin{aligned}
z(t) &= \mathbf{T}_t (x^0 + A^{-1} Bv^\infty) + \int_0^t \mathbf{T}_{t-s} Bw(s)\, ds \\
&= \mathbf{T}_t z^0 + \int_0^t \mathbf{T}_{t-s} Bw(s)\, ds, \qquad \forall\, t \in \mathbb{R}_+,
\end{aligned}$$

where $z(0) = z^0$. Consequently, for $s_0 \geq 0$,

$$z(t) = \mathbf{T}_{t-s_0} z(s_0) + \int_{s_0}^{t} \mathbf{T}_{t-s} B w(s)\, ds, \qquad t \geq s_0. \tag{A.26}$$

By infinite-time admissibility of B, there exists $\alpha > 0$ such that

$$\left\| \int_{s_0}^{t} \mathbf{T}_{t-s} B w(s)\, ds \right\| \leq \alpha \left(\int_{s_0}^{t} |w(s)|^2\, ds \right)^{1/2}, \qquad \forall\, t \geq s_0 \geq 0. \tag{A.27}$$

Let $\varepsilon > 0$. Since $w := v - v^\infty \vartheta \in L^2(\mathbb{R}_+, \mathbb{R})$, we obtain from (A.27) that there exists $s_0 \geq 0$ such that

$$\left\| \int_{s_0}^{t} \mathbf{T}_{t-s} B w(s)\, ds \right\| \leq \varepsilon/2, \qquad \forall\, t \geq s_0.$$

Finally, by strong stability, there exists $s_1 \geq 0$ such that

$$\|\mathbf{T}_t z(s_0)\| \leq \varepsilon/2, \qquad \forall\, t \geq s_1.$$

It follows from (A.26), that $\|z(t)\| \leq \varepsilon$ for all $t \geq s_0 + s_1$. Consequently, $\|x(t) + A^{-1} B v^\infty\| \leq \varepsilon$ for all $t \geq s_0 + s_1$. It is clear from the definition of z that if $v^\infty = 0$, we need no longer assume that $0 \in \mathrm{res}(A)$ in order to show that $\lim_{t\to\infty} \|x(t)\| = 0$. □

Proof of Lemma 9.1.5. Let $x^0 \in X$, $w \in F(\mathbb{Z}_+, \mathbb{R})$, $v = \mathcal{H} w$ and assume that $0 \in \mathrm{res}(A)$. The solution of (9.3a) is given by

$$x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-s} B v(s)\, ds, \qquad \forall\, t \in \mathbb{R}_+,$$

or, equivalently,

$$x(t) = \mathbf{T}_t x^0 + (Fv)(t), \qquad \forall\, t \in \mathbb{R}_+, \tag{A.28}$$

where $F : L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, X)$ is the shift-invariant operator defined by

$$(Fu)(t) := \int_0^t \mathbf{T}_{t-s} B u(s)\, ds, \qquad u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}).$$

Define $K : L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, X)$ by

$$(Ku)(t) := (F\mathscr{I}u)(t) + A^{-1} B (\mathscr{I}u)(t), \qquad u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}).$$

Recall from (7.12) that,

$$(\mathscr{I}(\mathcal{H}(\Delta v)))(t) = \tau(\mathcal{H}u)(t) - \tau u(0) + (t - n_t\tau)(\mathcal{H}(\Delta u))(t), \qquad \text{(A.29)}$$
$$\forall\, t \in \mathbb{R}_+,\ u \in F(\mathbb{Z}_+, \mathbb{R}),$$

where $\tau > 0$ denotes the sampling period and n_t denotes the integer part of t/τ. Set $h_c(t) := (t - n_t\tau)(\mathcal{H}(\Delta w))(t)$ for all $t \in \mathbb{R}_+$. By (A.29) it follows that,

$$\begin{aligned}(K\mathcal{H}\Delta w)(t) &= (F\mathscr{I}\mathcal{H}\Delta w)(t) + A^{-1}B(\mathscr{I}\mathcal{H}\Delta w)(t)\\ &= \tau(Fv)(t) - \tau(F\vartheta_c)(t)w(0) + (Fh_c)(t) + \tau A^{-1}Bv(t)\\ &\quad -\tau A^{-1}Bw(0) + A^{-1}Bh_c(t), \qquad \forall\, t \in \mathbb{R}_+. \end{aligned} \qquad \text{(A.30)}$$

Rearranging (A.30) we obtain,

$$\begin{aligned}(Fv)(t) &= \frac{1}{\tau}(K\mathcal{H}\Delta w)(t) + (F\vartheta_c)(t)w(0) - \frac{1}{\tau}(Fh_c)(t)\\ &\quad - A^{-1}Bv(t) + A^{-1}Bw(0) - \frac{1}{\tau}A^{-1}Bh_c(t), \qquad \forall\, t \in \mathbb{R}_+.\end{aligned}$$

Combining this with (A.28) and rearranging gives,

$$\begin{aligned}x(t) + A^{-1}Bv(t) =&\mathbf{T}_t x^0 + \frac{1}{\tau}(K\mathcal{H}\Delta w)(t) + (F\vartheta_c)(t)w(0)\\ &- \frac{1}{\tau}(Fh_c)(t) + A^{-1}Bw(0) - \frac{1}{\tau}A^{-1}Bh_c(t), \qquad \forall\, t \in \mathbb{R}_+.\end{aligned} \qquad \text{(A.31)}$$

A straightforward calculation shows that

$$F\vartheta_c = (\mathbf{T} - I)A^{-1}B,$$

and

$$(Ku)(t) = \int_0^t \mathbf{T}_{t-s}A^{-1}Bu(s)\,ds, \qquad \forall\, u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}).$$

Combining the above with (A.31) gives

$$\begin{aligned}x(t) + A^{-1}Bv(t) =&\mathbf{T}_t(x^0 + A^{-1}Bw(0)) + \frac{1}{\tau}\int_0^t \mathbf{T}_{t-s}A^{-1}B(\mathcal{H}\Delta w)(s)\,ds\\ &- \frac{1}{\tau}(Fh_c)(t) - \frac{1}{\tau}A^{-1}Bh_c(t), \qquad \forall\, t \in \mathbb{R}_+.\end{aligned} \qquad \text{(A.32)}$$

Note that since $\Delta w \in l^2(\mathbb{Z}_+, \mathbb{R})$, it follows that $\mathcal{H}(\Delta w) \in L^2(\mathbb{R}_+, \mathbb{R})$. Furthermore, since the function

$$t \mapsto t - n_t\tau = t - \tau\lfloor t/\tau \rfloor, \qquad t \in \mathbb{R}_+$$

is bounded (it takes values in $[0, \tau)$), it follows that $h_c \in L^2(\mathbb{R}_+, \mathbb{R})$ and moreover,

$$\lim_{t\to\infty} h_c(t) = 0. \tag{A.33}$$

To prove that $\lim_{t\to\infty} \|x(t) + A^{-1}Bv(t)\| = 0$, note that by (A.32), (A.33) and the strong stability of $\mathbf{T}$ it is sufficient to show that

$$\int_0^t \mathbf{T}_{t-s}A^{-1}B(\mathcal{H}\Delta w)(s)\, ds - (Fh_c)(t) \to 0, \qquad t \to \infty.$$

We first show that

$$\lim_{t\to\infty} \int_0^t \mathbf{T}_{t-s}A^{-1}B(\mathcal{H}\Delta w)(s)\, ds = 0. \tag{A.34}$$

Since,

$$\int_0^t \mathbf{T}_{t-s}A^{-1}B(\mathcal{H}\Delta w)(s)\, ds = A^{-1}\int_0^t \mathbf{T}_{t-s}B(\mathcal{H}\Delta w)(s)\, ds$$

and $\mathcal{H}\Delta w \in L^2(\mathbb{R}_+, \mathbb{R})$, an application of 9.1.4 shows that (A.34) holds. Noting that

$$(Fh_c)(t) = \int_0^t \mathbf{T}_{t-s}Bh_c(s)\, ds,$$

and $h_c \in L^2(\mathbb{R}_+, \mathbb{R})$, again applying Lemma 9.1.4, we see that

$$\lim_{t\to\infty} (Fh_c)(t) = 0.$$

Hence we have that $\lim_{t\to\infty} \|x(t) + A^{-1}Bv(t)\| = 0$. □

Remark 12.1.8. The proof of Lemma 9.1.5 is inspired by similar arguments to those used to prove Lemma 5.2 of [38]. The result in [38] differs from Lemma 9.1.5 in that, the input v is assumed to be in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ with derivative $v' \in L^2(\mathbb{R}_+, \mathbb{R})$, whereas, we make no assumptions about the derivative of v, or the integrability of v. ◇

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